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VOL. XV. JANUARY, 1931. No. 211.

NAPIER'S METHOD AS A BASIS FOR THE THEORY OF LOGARITHMS.

BY W. C. FLETCHER, M.A.

THE customary school method of dealing with logarithms (*i.e.* the initial definition: if $a^x=N$ then . . .) has one great defect: it gives no lead whatever towards the higher theory of logarithms, nor even towards methods for calculating them. "How shall we introduce our sixth form girls to the theory?" a mistress recently asked; "it seems to come like a bolt from the blue." Klein has some true and forcible things to say on the subject [*Elementar Mathematik vom höheren Standpunkte aus*, (3 ed. 1924) I, p. 155 *et seq.*] from among which a sentence or two may be quoted.

"This definition of $e=\mathrm{Lt}\left(1+\frac{1}{n}\right)^n$ is put at the head of the development without explanation as to why this remarkable limit is chosen and the resulting logarithms called *natural*." "Again the series development is often taken without introduction: assume $\log(1+x)=a_0+a_1x\ldots$, calculate the coefficients, etc. It remains quite unexplained how the very possibility of development in series should be suspected, especially with a *function so arbitrarily concocted as is the logarithm according to the school definition*."

In recent years the methods Klein refers to have been or are being abandoned, but the initial definition put in their place, viz.

$$\log x=\int_1^x \frac{du}{u}$$

also seems at first sight to have no connection with logarithms as previously understood. Klein goes on to point out that the best way to penetrate "to the inner connections . . . and to a full understanding of the theory of logarithms" is to follow the main lines of their historical development—in short, to go back to Napier.

The purpose of this supplement to Dr. Henderson's article * is to show that, and how, this can be done under school conditions.

It is becoming a common practice to introduce boys to logarithms by putting the tables into their hands and showing how they work, without any reference to indices or any other theoretical matter. The question arises sooner or later, "How are the tables made?" The first step has already been gained—to add something to a logarithm corresponds to applying to the number a corresponding factor; or if such language has now become familiar—as the logarithm increases in arithmetical progression, the number increases in geometric progression. This was Napier's fundamental idea.

* *Gazette*, vol. xv, p. 250, December, 1930.

The first step then towards calculation of logarithms is to form a series of numbers in geometric progression, but they must be close packed, *i.e.* the factor must be close to unity.

It is more convenient to work with two factors, as follows:

Make a table of numbers (it will contain 8 columns of 20 rows) in which the heads of the columns are powers of 1·1, *i.e.*

$$1 \quad 1{\cdot}1 \quad 1{\cdot}21 \quad 1{\cdot}331 \ldots 1{\cdot}948717$$

(6 decimal places are sufficient).

In each column proceed by the factor 1·005, *i.e.* add 1/200 part to the preceding number.

So the 3rd column for instance begins 1·21

1·216050

1·222130

.....

and ends 1·330272.

The 8 columns go on to a single sheet of foolscap and can be calculated with little labour and verified by working crosswise with the factor 1·1.

We now have a series of about 150 numbers extending from 1 to 2 and all of the form $(1{\cdot}1)^{c-1} \times (1{\cdot}005)^{r-1}$, where c is the number of the column, and r of the row.

Accordingly, if we know the two "key logs", viz. log 1·1 and log 1·005, we can get the logarithm of any number in the table. The original table with the logarithms inserted Napier calls the Radical Table.

The process of getting these key logarithms is the same as that of interpolating for numbers not in the table, and perhaps it is easier to grasp the latter application first. So for the moment we will assume the values of the key logs, viz.

$$\log 1{\cdot}1 = 0{\cdot}0953102, \qquad \log 1{\cdot}005 = 0{\cdot}00498754.$$

Suppose we want log 1·5. The nearest tabular number is 1·501070, the 6th number in the 5th column; its logarithm is therefore

$$\begin{array}{rl} 0{\cdot}0953102 \times 4 & = 0{\cdot}3812408 \\ +0{\cdot}00498754 \times 5 & \underline{0{\cdot}0249377} \\ & 0{\cdot}406178- \end{array}$$

dropping the 7th figure as unreliable.

To get log 1·5 we have to correct this for an excess of 0·001070. Here Napier's working principle comes in. It is set forth much in Napier's form in Dr. Henderson's paper; but it can be put more simply and must be so put if it is to be available for ordinary school use. The following is an attempt at a simpler statement.

If we have a series of numbers in geometric progression, their successive differences $ar - a$, $ar^2 - ar$, ..., form another geometric progression *with the same factor as the original*, *i.e.* the successive differences are proportional to the successive terms.

Let us represent the connection between number and logarithm by a distance-time graph, taking the arithmetically growing logarithm or abscissa as the "time" and the geometrically growing number or ordinate as "distance". It follows from what has just been said that the "velocity" is always *proportional* to the ordinate.

If we not only take, as we must, the ordinate as unity when the abscissa or "time" is zero, but also take with Napier the velocity at that time as unity, we have the velocity always *equal* to the ordinate.

If then P, Q are two points on the graph, P preceding Q, the velocity at P is the least and that at Q the greatest in the interval PQ, *i.e.* the "time" in the interval *lies between* the two values obtained by dividing the increase of the ordinate by the ordinates at P and Q respectively.

In other words, reverting to our numerical example, the correction in the logarithm required lies between $0{\cdot}001070 \div 1{\cdot}5$ and $0{\cdot}001070 \div 1{\cdot}501070$, *i.e.* between 0·0007133 and 0·0007127 ;

$$\begin{aligned} \therefore\ \log 1{\cdot}501070 &= 0{\cdot}406178 \\ \text{correction for } 1070 \quad & \quad 713 \\ \log 1{\cdot}5 \quad &= 0{\cdot}405465 \end{aligned}$$

In this case the result is in fact correct to the 6th figure—more than can usually be expected.

As another example, let us find log 2.

The nearest tabular number is 1·997924, the 6th number in the 8th column,

$$\begin{aligned} \therefore\ \log 1{\cdot}997924 &= 0{\cdot}0953102 \times 7 \quad = 0{\cdot}6671714 \\ &\quad + 0{\cdot}00498754 \times 5 \quad 0{\cdot}0249377 \\ & \qquad\qquad\qquad\quad 0{\cdot}692109- \end{aligned}$$

The correction for 0·002076 lies between $0{\cdot}002076 \div 2$ and $0{\cdot}002076 \div 1{\cdot}998$, *i.e.* between 0·001038 and 0·001039 ;

$$\begin{aligned} \therefore\ \log 2 &= 0{\cdot}692109 \\ &\quad + \quad 1038+ \\ & \qquad\quad 0{\cdot}69315 \end{aligned}$$

This is correct to the 5th figure, but doubling to get log 4 and trebling to get log 8 it is clear that their values are 1·38629 and 2·07944.

To get log 10, first calculate log 1·25 and add log 8. The result obtained is 2·30259, which again is correct to the 5th decimal.

We can now consider the formation of the key logarithms.

First find that of 1·0001. The logarithm of unity is zero and the correction for 0·0001 lies between 0·0001 and $0{\cdot}0001 \div 1{\cdot}0001$, *i.e.* between 0·0001 and 0·00009999. Here Napier says " we may take either, or anything between or ' best of all ' 0·000099995 ".

Accordingly we take $\log (1{\cdot}0001)^{50}$ as 0·00499975.

Using the binomial expansion, or forming a supplementary table, as did Napier, not knowing the binomial, this gives

$$\log 1{\cdot}005012269 = 0{\cdot}00499975.$$

Correcting as before for 1227, we get $\log 1{\cdot}005 = 0{\cdot}00498754$.

To get log 1·1 : the last number of the first column of the main table, viz. 1·099398 is the 19th power of 1·005 and its logarithm is 0·094763. Correcting for 0·000602 we get finally $\log 1{\cdot}1 = 0{\cdot}0953102$. Working on these lines the labour involved in calculating a considerable part of the ordinary 4 or 5 figure tables of hyperbolic logarithms is not serious, and for those numbers at all events that lie between 1 and 2 the results are accurate to 5 figures.

Two main points leading on to the higher theory emerge.

First, as the process becomes familiar it becomes clear that it is in fact the process of integrating, and that $\log y = \int_1^y \frac{du}{u}$.

Second, Napier has no base, but the question arises what is the number whose logarithm is unity. There is, of course, no difficulty in finding it empirically, say, by extending the table till the unit logarithm is reached. The 10th number in the 11th column is 2·712824 and its log 0·99799. We have to correct the *number* for a defect 0·00201 in the logarithm, and the correction will be about, but greater than, $0{\cdot}00201 \times 2{\cdot}7128$, *i.e.* 0·005453, making the number $> 2{\cdot}718277$. But taking the number as 2·718 the correction would be less than 0·005463, making the number $< 2{\cdot}718287$, so we clearly have 2·71828 correct to 5 places.

But the question remains what is the real nature of the number to which this is an approximation.

The fundamental assumption was that

$$\log\left(1+\frac{1}{10^4}\right) \text{ lies between } \frac{1}{10^4} \text{ and } \frac{1}{10^4}\div\left(1+\frac{1}{10^4}\right)$$

i.e. between $$\frac{1}{10^4} \text{ and } \frac{1}{10^4+1};$$

thus antilog 1 lies between $\left(1+\frac{1}{10^4}\right)^{10^4}$ and $\left(1+\frac{1}{10^4}\right)^{10^4+1}$.

Similarly, if we had started with $1+\frac{1}{10^7}$ we should have found antilog 1 between $\left(1+\frac{1}{10^7}\right)^{10^7}$ and $\left(1+\frac{1}{10^7}\right)^{10^7+1}$. Do these two results agree? Do the latter pair of bounds lie between the former? In other words, is it true that as n increases $\left(1+\frac{1}{n}\right)^n$ increases, but $\left(1+\frac{1}{n}\right)^{n+1}$ decreases?

As we are concerned only with integral values of n we can use the binomial theorem without fear, and the truth of the former statement is obvious. The question whether $\left(1+\frac{1}{n}\right)^{n+1}$ decreases with n is more delicate, but is still easily answered by ordinary algebra.

We have then a very good basis for the assumption that $\left(1+\frac{1}{n}\right)^n$ has a limit, whatever further investigation may be required when n is not an integer, and that this limit has the approximate value 2·71828 and that it is in fact the base of the system of logarithms we have been constructing.

Two other points from the subsequent seventeenth century development are worthy of attention—the discovery of the two series.

N. Mercator, some fifty years after Napier, in effect said (I follow Klein as my authority):

$$\log(1+x)=\int_0^x \frac{dx}{1+x}$$

$$=\int_0^x (1-x+x^2-\ldots)dx$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots.$$

This will need subsequent examination and criticism but it is the natural, as it is the historical, point of departure.

Newton got hold of this result and by reversion of series found that if

$$y=\log x, \text{ then } x=1+y+\frac{y^2}{2!}+\ldots.$$

Again the most natural introduction to the matter.

To get logarithms to base 10 there is no need to adopt a fresh method. Neither Briggs nor Napier appears to have noticed that they could be derived from Napier's, an oversight natural enough with the cumbersome presentment inevitable in a first discovery. But to us it is obvious even apart from our knowledge of the rule for changing the base, that all that is required is to slow down the rate of uniform growth in the logarithm so that the logarithm reaches 1 when the number is 10, *i.e.* to divide by log 10.

It is some considerable saving of labour to reconstruct the Radical Table, using the key logs to base 10. It is then only necessary in each individual case to transform the corrections.

W. C. FLETCHER.

FORMULÆ FOR POWERS AND REVERSION OF SERIES.

By Albert Eagle, B.Sc.

The formulæ for the powers and the reversion of any given series are not so readily accessible to those who are dependent upon modern mathematical text-books as they should be considering the frequent and manifold uses to which such formulæ can be put. I thought therefore that the following compendium of such formulæ would be useful to readers of the *Mathematical Gazette*. The chief formulæ in this paper are given in De Morgan's *Differential and Integral Calculus*, but this book must be getting increasingly unavailable to ordinary readers except in reference libraries.

The formulæ in question are useless if not accessible, since their derivation is too tedious to effect when they are required.

I. Formulæ for Powers of Series.

Let us require the first several terms of the nth power of any given series. If the first term is not unity, division will make it so, so we will suppose that

$$u \equiv 1 + ax + bx^2 + cx^3 + dx^4 + ex^5 + \dots ; \tag{1}$$

and that we require to determine the coefficients in the expansion

$$u^n \equiv 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots . \tag{2}$$

As a simple change of variable will make $a=1$ in (1) and simplify the coefficients in (2) we will suppose this done. It will be convenient to denote the series

$$a + bx + cx^2 + dx^3 + \dots \tag{3}$$

by v; and to indicate this series in the particular case when $a=1$ by v_1. Let us further, for brevity, denote the binomial coefficients, $n/1!$, $n(n-1)/2!$, ... by n_1, n_2, ... Then we have

$$\begin{aligned} \{1 + x + bx^2 + \dots\}^n &\equiv (1 + xv_1)^n \\ &= 1 + n_1 xv_1 + n_2 x^2 v_1^2 + n_3 x^3 v_1^3 + \dots . \end{aligned} \tag{4}$$

Now the series for v_1 and v_1^2 are easily written down. To obtain the series for v_1^3, v_1^4 ... we must write

$$v_1 \equiv 1 + bx + cx^2 + \dots \equiv 1 + xw ;$$

where $w \equiv b + cx + dx^2 + \dots ;$

and further write $w \equiv b + xs ;$

where $s \equiv c + dx + \dots .$

This being so, the terms up to x^8 inclusive give :

$$\begin{aligned} u^n = 1 &+ n_1 xv_1 + n_2 x^2 v_2 \\ &+ n_3 x^3 \{1 + 3xw + 3x^2 w^2 + x^3(b^3 + 3b^2 xs + 3bx^2 s^2)\} \\ &+ n_4 x^4 \{1 + 4xw + 6x^2 w^2 + 4x^3(b^3 + 3b^2 xs) + x^4 b^4\} \\ &+ n_5 x^5 (1 + 5xw + 10x^2 w^2 + 10x^3 b^3) \\ &+ n_6 x^6 (1 + 6xw + 15x^2 b^2) + n_7 x^7 (1 + 7xw) + n_8 x^8 . \end{aligned}$$

Replacing v_1, w and s by their series, and v_1^2, w^2 and s^2 by

$$\begin{aligned} v_1^2 = 1 &+ 2bx + (b^2 + 2c)x^2 + (2bc + 2d)x^3 \\ &+ (c^2 + 2bd + 2e)x^4 + (2cd + 2be + 2f)x^5 \\ &+ (d^2 + 2ce + 2bf + 2g)x^6 ; \end{aligned}$$

$$v^2 = b^2 + 2bcx + (c^2 + 2bd)x^2 + (2cd + 2be)x^3 ;$$

and $$s^2 = c^2 ;$$

and collecting the coefficients of like powers of x we get, for the coefficients in (2), (when $a = 1$),

TABLE I.

$$A = n_1$$
$$B = n_1 b + n_2$$
$$C = n_1 c + 2n_2 b + n_3$$
$$D = n_1 d + 2n_2(c + \tfrac{1}{2}b^2) + 3n_3 b + n_4$$
$$E = n_1 e + 2n_2(d + bc) + 3n_3(c + b^2) + 4n_4 b + n_5$$
$$F = n_1 f + 2n_2(e + bd + \tfrac{1}{2}c^2) + 3n_3(d + 2bc + \tfrac{1}{3}b^3) + 4n_4(c + \tfrac{3}{2}b^2) + 5n_5 b + n_6$$
$$G = n_1 g + 2n_2(f + be + cd) + 3n_3(e + 2bd + c^2 + b^2 c) + 4n_4(d + 3bc + b^3) + 5n_5(c + 2b^2) + 6n_6 b + n_7$$
$$H = n_1 h + 2n_2(g + bf + ce + \tfrac{1}{2}d^2) + 3n_3(f + 2be + 2cd + b^2 d + bc^2) + 4n_4(e + 3bd + \tfrac{3}{2}c^2 + 3b^2 c + \tfrac{1}{4}b^4) + 5n_5(d + 4bc + 2b^3) + 6n_6(c + \tfrac{5}{2}b^2) + 7n_7 b + n_8.$$

It is worth noting that the numerals in front of the n's cancel the last factor of their denominators.

The coefficients when $a \neq 1$ are easily obtained from this table. We have only to suppose that a, b, c, ... have dimensions of 1, 2, 3, ... respectively and introduce a's into every term till all terms in the expression for any coefficient have the same dimension as the first term. Thus when $a \neq 1$, we have

$$D = n_1 d + 2n_2(ac + \tfrac{1}{2}b^2) + 3n_3 a^2 b + n_4 a^4 ;$$

and so on.

When we come to the reversion of series we shall require to have handy for reference the coefficients of the expansion of

$$v^n \equiv (a + bx + cx^2 + \ldots)^n. \qquad (5)$$

Denoting this expansion, in the case when $a = 1$, by

$$1 + A_n x + B_n x^2 + C_n x^3 + \ldots ; \qquad (6)$$

it is obvious that the new coefficients may be obtained from Table I by introducing the a's in that table as above described and then changing a into b; b into c; c into d; and so on.

Hence we have

TABLE II.

$$A_n = n_1 b$$
$$B_n = n_1 c + n_2 b^2$$
$$C_n = n_1 d + 2n_2 bc + n_3 b^3$$
$$D_n = n_1 e + 2n_2(bd + \tfrac{1}{2}c^2) + 3n_3 b^2 c + n_4 b^4$$
$$E_n = n_1 f + 2n_2(be + cd) + 3n_3(b^2 d + bc^2) + 4n_4 b^3 c + n_5 b^5$$
$$F_n = n_1 g + 2n_2(bf + ce + \tfrac{1}{2}d^2) + 3n_3(b^2 e + 2bcd + \tfrac{1}{3}c^3) + 4n_4(b^3 d + \tfrac{3}{2}b^2 c^2) + 5n_5 b^4 c + n_6 b^6.$$

If a in (5) is not equal to unity we have only to introduce a's into all the terms in Table II till each term is of the nth degree in the coefficients. Thus when $a \neq 1$, we have

$$C_n = n_1 d a^{n-1} + 2n_2 bc a^{n-2} + n_3 b^3 a^{n-3}$$

and so on.

It may be observed that Tables I and II hold for all values of n, positive or negative, integral or fractional.

It is clear on looking at equation (4), that if instead of getting the expansion of $(1+xv)^n$ we had wanted the expansion of any arbitrary function of xv, say $\phi(xv)$, where

$$\phi(z) \equiv \phi_0 + \phi_1 z + \phi_2 z^2 + \phi_3 z^3 + \dots \qquad (7)$$

in which ϕ_p is an abbreviation for $\phi^{(p)}(0)/p!$, we should have got (in the case when $a=1$) exactly the same coefficients as are given in Table I with the n's replaced by the ϕ's from (7). It is not necessary to repeat Table I for the sake of such an obvious modification.

Five of the most frequently wanted cases are $(1+xv)^{\frac{1}{2}}$, $(1+xv)^{-\frac{1}{2}}$, $(1+xv)^{-1}$, $\log(1+xv)$ and e^{xv} or $\exp(xv)$. To facilitate rapid working the n's for these cases are here tabulated in

TABLE III.

n	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{8}$	$+\frac{1}{16}$	$-\frac{5}{128}$	$+\frac{7}{256}$	$-\frac{21}{1024}$	$+\frac{33}{2048}$	$-\frac{429}{32768}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{3}{8}$	$-\frac{5}{16}$	$+\frac{35}{128}$	$-\frac{63}{256}$	$+\frac{231}{1024}$	$-\frac{429}{2048}$	$+\frac{6435}{32768}$
-1	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$
log	1	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{4}$	$+\frac{1}{5}$	$-\frac{1}{6}$	$+\frac{1}{7}$	$-\frac{1}{8}$
exp	$\frac{1}{1!}$	$\frac{1}{2!}$	$\frac{1}{3!}$	$\frac{1}{4!}$	$\frac{1}{5!}$	$\frac{1}{6!}$	$\frac{1}{7!}$	$\frac{1}{8!}$

Perhaps *the* most frequently required expansion is

$$\sqrt{(1+x^2v^2)} \equiv \sqrt{\{1+(ax+bx^2+cx^3+\dots)^2\}}. \qquad (8)$$

Here we have $n_1=0=n_3=n_5=n_7$ and

$$n_2=\tfrac{1}{2}\,;\quad n_4=-\tfrac{1}{8}\,;\quad n_6=+\tfrac{1}{16},\ \text{and}\ n_8=-\tfrac{5}{128}\,;$$

whence Table I gives, on inserting the a's as above described, for the expansion of (8),

$$\begin{aligned} &1+\tfrac{1}{2}a^2x^2+abx^3+(ac+\tfrac{1}{2}b^2-\tfrac{1}{8}a^4)x^4 \\ &+(ad+bc-\tfrac{1}{2}a^3b)x^5+(ae+bd+\tfrac{1}{2}c^2-\tfrac{1}{2}a^3c-\tfrac{3}{8}a^2b^2+\tfrac{1}{16}a^6)x^6 \\ &+(af+be+cd-\tfrac{1}{2}a^3d-\tfrac{3}{2}a^2bc-\tfrac{1}{2}ab^3+\tfrac{3}{8}a^5b)x^7 \\ &+(ag+bf+ce+\tfrac{1}{2}d^2-\tfrac{1}{2}a^3e-\tfrac{3}{2}a^2bd-\tfrac{3}{2}a^2c^2 \\ &\quad -\tfrac{3}{2}ab^2c-\tfrac{1}{8}b^4+\tfrac{3}{8}a^5c+\tfrac{1}{16}a^4b^2-\tfrac{5}{128}a^8)x^8+\dots\,. \end{aligned} \qquad (9)$$

If it is desired to obtain any power of a *Fourier series* a new variable, ρ, must be introduced and the nth harmonic terms multiplied by ρ^n. The desired power is then obtained as if the series was a power series in ρ. Replacing the sines and cosines by their exponential values will generally facilitate this. After the expansion ρ is replaced by unity. The resulting series can be rearranged as a Fourier series, if desired, but the coefficient of any harmonic will itself be an infinite series in the original Fourier coefficients.

II. FORMULÆ FOR THE REVERSION OF SERIES.

If y is connected with x by the relation

$$y=q+ax+bx^2+\dots$$

x may be expanded in powers of $y-q$. As this is absorbing q into y we may suppose without loss of generality that $q=0$. As before, it will be convenient

to tabulate the coefficients in the simpler forms which they take when $a=1$. Hence we will suppose that when

$$y=x+bx^2+cx^3+dx^4+\dots, \tag{10}$$

$$x=y+\mathbf{B}y^2+\mathbf{C}y^3+\mathbf{D}y^4+\dots. \tag{11}$$

Now the solution of this problem is contained in Lagrange's theorem in the differential calculus; and it is a remarkable fact that the coefficients **B, C,** ... are expressible in terms of the coefficients A, B, C, ... of Table II. By Lagrange's theorem, if

$$x=k+y\phi(x) \tag{12}$$

we have, for the expansion of any function, $F(x)$ of x,

$$F(x)=F(k)+\frac{y}{1}\phi(k)F'(k)+\frac{y^2}{2}\cdot\frac{1}{1!}\frac{d}{dk}[\{\phi(k)\}^2F'(k)]$$

$$+\frac{y^3}{3}\cdot\frac{1}{2!}\frac{d^2}{dk^2}[\{\phi(k)\}^3F'(k)]+\dots. \tag{13}$$

If in (12) we write $\phi(x)=(1+bx+cx^2+\dots)^{-1}$ and put $k=0$, that equation becomes identical with (10), while (13) then gives, in the special case when $F(x)\equiv x$,

$$x=y+\frac{y^2}{2}A_{-2}+\frac{y^3}{3}B_{-3}+\frac{y^4}{4}C_{-4}+\dots \tag{14}$$

on using the notation for the coefficients in Table II. Comparing (11) and (14) we have

$$\mathbf{B}=\tfrac{1}{2}A_{-2};\ \mathbf{C}=\tfrac{1}{3}B_{-3};\ \mathbf{D}=\tfrac{1}{4}C_{-4}; \tag{15}$$

These relations show that the values of the n's to be inserted in Table II to turn the A_n, B_n, ... into **B, C,** respectively are given by the following scheme:

-1						(for A_n line)
-1	$+2$					(„ B_n „)
-1	$+2\frac{1}{2}$	-5				(„ C_n „)
-1	$+3$	-7	$+14$			
-1	$+3\frac{1}{2}$	$-9\frac{1}{3}$	$+21$	-42		
-1	$+4$	-12	$+30$	-66	$+132$.	

Hence the following table for the coefficients **B, C,** ....

TABLE IV.

$$\mathbf{B}=-b$$

$$\mathbf{C}=-c+2b^2$$

$$\mathbf{D}=-d+5bc-5b^3$$

$$\mathbf{E}=-e+6(bd+\tfrac{1}{2}c^2)-21b^2c+14b^4$$

$$\mathbf{F}=-f+7(be+cd)-28(b^2d+bc^2)+84b^3c-42b^5$$

$$\mathbf{G}=-g+8(bf+ce+\tfrac{1}{2}d^2)-36(b^2e+2bcd+\tfrac{1}{3}c^3)+120(b^3d+1\tfrac{1}{2}b^2c^2)-330b^4c+132b^6.$$

If the first coefficient in (10) is a and not unity the correct values of **B, C,** ... can be obtained from Table IV by inserting a's into all the terms till all the

terms on the pth line are of degree p in the coefficients a, b, c, ... and then dividing the whole line by a^{2p+1}. Thus when $a \neq 1$, we have

$$\mathbf{D} = (-a^3d + 5abc - 5b^3)/a^7,$$

and so on.

We may sometimes desire to reverse a series which is not of the form of that in (10). Thus we may require to reverse

$$y = x^m\{1 + bx^n + cx^{2n} + \ldots\}. \tag{16}$$

By a change of variable this may be written

$$y = x\{1 + bx^p + cx^{2p} + \ldots\}, \tag{17}$$

where p need not be an integer.

As expanding x in (17) in powers of y is equivalent to expanding x^m in (16), and this may not be what is wanted, the general expansion of x^q in (17) in powers of y will be given.

Assuming both p and q are integers the required expansion can be obtained from (13) by taking $F(x) \equiv x^q$. The general term in (13) now becomes

$$\frac{y^s}{s!}\left(\frac{d}{dk}\right)^{s-1}\{qk^{q-1}(1 + A_{-s}k^p + B_{-s}k^{2p} + \ldots)\}_{k=0},$$

on writing out the expansion of $\{\phi(k)\}^s$. But this is obviously zero unless $s = q$, or $p+q$, or $2p+q$, etc., when the term becomes

$$y^q, \text{ or } \frac{qy^{p+q}}{p+q}A_{-p-q}, \text{ or } \frac{qy^{2p+q}}{2p+q}B_{-2p-q}, \text{ etc.},$$

respectively. Hence the reversion of (17), for the qth power of x, is

$$x^q = y^q + \frac{qy^{p+q}}{p+q}A_{-p-q} + \frac{qy^{2p+q}}{2p+q}B_{-2p-q} + \ldots. \tag{18}$$

Denoting this series by

$$x^q = y^q + \mathfrak{B}y^{p+q} + \mathfrak{C}y^{2p+q} + \ldots,$$

the following scheme of n's inserted in Table II will turn the A, B, C, ... of that table into the present $\mathfrak{B}$, $\mathfrak{C}$, ... :

$$-q$$

$$-q, \quad +\frac{q(2p+q+1)}{2}$$

$$-q, \quad +\frac{q(3p+q+1)}{2}, \quad -\frac{q(3p+q+1)(3p+q+2)}{2\,.\,3}$$

$$-q, \quad +\frac{q(4p+q+1)}{2}, \quad -\frac{q(4p+q+1)(4p+q+2)}{2\,.\,3}, \quad +\ldots.$$

.....

The reversion in (18) has been obtained on the assumption that p and q are integers: the method of proof is meaningless when p and q are not integers. Nevertheless in this case an expansion of x^q in terms of y could be obtained from (17) by assuming an expansion with unknown coefficients, raising it to the various powers required, and substituting in (17), when the unknown coefficients could be obtained by equating coefficients of like powers of y on both sides of (17). Were this done it is obvious that the unknown coefficients would be obtained as polynomials in the two variables p and q. Now the coefficients $\mathfrak{B}$, $\mathfrak{C}$, ... as above obtained are polynomials in p and q, and they are correct for all positive integral values of p and q. As it is impossible for two different polynomials (even in two variables) to be identical for all positive

integral values of the variables, it follows that the coefficients $\mathfrak{B}$, $\mathfrak{C}$, ... are correct for all positive or negative values of p and q. Equation (17) is not convergent at $x=0$ when p is negative, nor (probably) near there, and the same is true of (18) at $y=0$; but that is the only difference that arises when p is negative.

The two most important cases of the reversion of (17) required in practice are when $p=2$ and $p=\frac{1}{2}$. These reversions are so useful that the formulæ for them are here given to the first six terms.

When $$y=x+bx^3+cx^5+dx^7+\dots$$
the reversion is

$$\begin{aligned} x=y&-by^3+(3b^2-c)y^5-(12b^3-8bc+d)y^7\\ &+(55b^4-55b^2c+10bd+5c^2-e)y^9\\ &-\{91(3b^5-4b^3c)+6(13bc^2+13b^2d-2cd-2be)+f\}y^{11}+\dots . \end{aligned} \tag{19}$$

When $$y=x^2+bx^3+cx^4+\dots , \tag{20}$$
the expansion of x^2 (which is obtained by taking $p=\frac{1}{2}$ and $q=1$) is

$$\begin{aligned} x^2=y&-by^{\frac{3}{2}}+(\tfrac{3}{2}b^2-c)y^2-(\tfrac{21}{8}b^3-\tfrac{7}{4}bc+d)y^{\frac{5}{2}}\\ &+(5b^4-10b^2c+4bd+2c^2-e)y^3\\ &-(\tfrac{1287}{128}b^5-\tfrac{429}{16}b^3c+\tfrac{99}{8}(b^2d+bc^2)-\tfrac{9}{8}(be+cd)+f)y^{\frac{7}{2}}+\dots . \end{aligned} \tag{21}$$

Taking $p=\frac{1}{2}$ and $q=\frac{1}{2}$ gives us the expansion

$$\begin{aligned} x=\sqrt{y}\{1&+(\tfrac{5}{8}b^2-\tfrac{1}{2}c)y+(\tfrac{715}{128}b^4-\tfrac{143}{16}b^2c+\tfrac{7}{4}bd+\tfrac{7}{8}c^2-\tfrac{1}{2}e)y^2+\dots\}\\ &-\{\tfrac{1}{2}by+(b^3-\tfrac{3}{2}bc+\tfrac{1}{2}d)y^2\\ &+(\tfrac{7}{2}b^5-10b^3c+5b^2d+5bc^2-2be-2cd+\tfrac{1}{2}f)y^3+\dots\}. \end{aligned} \tag{22}$$

The loci obtained by giving the $\pm$ sign to the fractional powers of y in (21) or to the $\sqrt{y}$ in (22) are the two arcs into which the part of the curve represented by (20) which is near the origin is divided by the origin; since on each arc only one value of x is given for one value of y, the series can not converge for a greater value of y than the smaller of the two maxima between which the minimum at the origin necessarily * lies.

If k is the upper limit of the values of y for which (21) and (22) converge, the curve represented by these series has two points on every line parallel to the x-axis between $y=0$ and $y=k$; it is clear that the expression in the second brackets $\{\}$ in (22) represents the curvilinear diameter of this curve, while the part preceding it represents half the width of the curve. A. Eagle.

APPENDIX.

Note on Ramanujan's method of solving equations.

Ramanujan's method of solving the equation

$$1=A_1x+A_2x^2+A_3x^3+\dots$$
described by Dr. Wilson in the May issue of the *Mathematical Gazette* may be applied equally easily to the solution of the equation

$$y=x+bx^2+cx^3+\dots \qquad\qquad (A)$$
for x when x is small.

* Here the language is being used which allows a maximum to be infinite. And unless both the neighbouring maxima are infinite, or in crude phrase unless there are no maxima at all, the reversed series can not be convergent for all values of y. That this obvious necessary condition is not sufficient is shown by the case of $y=x^3+x^4$, for which the reversed series are not convergent for $y>\frac{1}{4}$. But in the case when $y>\frac{1}{4}$ we can get a convergent series for x in ascending powers of $y^{-\frac{1}{3}}$ (all multiplied by $y^{\frac{1}{3}}$) by writing the equation in the form $y=x^4(1+x^{-1})$ and taking $p=-\frac{1}{3}$ and $q=\frac{1}{3}$. For any parabola of even, finite degree a series for x, convergent for large values of y, can always be obtained in this manner; the two series may, however, leave a large intermediate region of y in which neither of them is convergent.

Ramanujan's original argument (applied to this case) was probably as follows: Divide (A) by successive powers of x and neglect the remaining positive powers. We get the equations

$$y/x=1 \text{ ..... (i)}; \quad y/x^2=1/x+b \text{ ..... (ii)};$$

$$y/x^3=1/x^2+b/x+c \text{ ..... (iii)},$$

and so on.

Denote the value of y/x in (i) by P_1; the value of y/x^2 in (ii) with P_1/y for $1/x$ by P_2; the value of y/x^3 in (iii) with P_2/y for $1/x^2$ and P_1/y for $1/x$ by P_3; and so on. We then have the equations

$$P_1=1; \quad P_2=P_1/y+b; \quad P_3=P_2/y^2+bP_1/y+c;$$

and so on with the value of x given by $\lim P_n/P_{n+1}$.

On working out the values of these P's it was at once seen that they were the coefficients of Table I with n_1 replaced by unity, n_2 by $1/y$, n_3 by $1/y^2$, etc. As explained above, the function whose expansion would give these values for the n's is easily found to be

$$\frac{z}{1}+\frac{z^2}{y}+\frac{z^3}{y^2}+\ldots \text{ or } \frac{yz}{y-z}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(B)$$

where z stands for $x+bx^2+\ldots$.

Now the series (B) is absolutely convergent if $|z/y|<1$. Let z be expressed in terms of x and the series rearranged in powers of x. But the expansion in powers of x is, by Table I,

$$P_1x+P_2x^2+P_3x^3+\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C)$$

and *this* is convergent when $\lim |xP_{n+1}/P_n|<1$. As it is easily seen that this condition must be the same as $|z/y|<1$, we see that *when* $z=y$, i.e. *when* $y=x+bx^2+\ldots$, *then* $x=\lim P_n/P_{n+1}$, where the P's are given by the successive lines of Table I with $n_1=1$; $n_2=1/y$, etc.

Thus the above paper, apart from showing that the Ramanujan P's are merely particular cases of the general universal coefficients tabulated in Table I, from which Table the first eight P's can be at once written down, also shows the place of Ramanujan's method in the general problem of determining x in terms of y when y is given by a series of powers of x. Further Ramanujan's method has drawn my attention to the fact (easily otherwise proved) that if all the suffixes to the n's in Table I are turned into indices the coefficients are connected by the recurrence relations:

$$A=n; \quad B=n(b+A); \quad C=n(c+bA+B);$$
$$D=n(d+cA+bB+C); \quad E=n(e+dA+cB+bC+D), \text{ etc.}$$

The high accuracy often attained even by the early Ramanujan approximations is remarkable, as any one can see by obtaining the approximations to $x\,(=\log(1+y))$ from the series

$$y=x+x^2/2\,!+x^3/3\,!+\ldots.$$

The P_4/P_5 approximation gives, when $y=2$, $\log_e 3=1{\cdot}10497$, an error of only $0{\cdot}58\%$, while the power series is divergent if $|y|>1$.

GLEANINGS FAR AND NEAR.

795. " and see how Mathematick rideth as a queen
cheer'd on her royal progress thru'out nature's realm."
—Robert Bridges, *The Testament of Beauty*, IV. 856-7
[*Per* Miss M. O. Stephens].

THE BRITISH ASSOCIATION MEETING, BRISTOL, 1930.

By Patrick Du Val.

The 100th meeting of the British Association was unquestionably one of the very greatest interest to mathematicians of almost all kinds; though by a conjunction of accidents straightforward algebraic geometry found itself almost completely left out of the programme. Interesting as were Mr. Richmond's investigations as to the possibility of expressing a given number as the sum of three cubes, many of his hearers must have regretted the loss of his very valuable researches on the Canonical Curve of genus 5, which he apparently decided at the last moment were not at a sufficiently advanced stage to form conveniently the subject for his paper. Other branches of geometry were represented by Mr. Coxeter, who gave an account of the modern analytical methods of discovering and completely enumerating the regular polytopes (a subject not as much studied in this country as one might wish), and by Mr. Hodge, who explained the work which has been done towards the use of topology for the discussion of algebraic surfaces, a surface being represented by a "Riemann fourfold" in the same way as a curve may be by a Riemann surface. Analysis, on the other hand, was exhaustively (to a simple geometer almost exhaustingly) treated. Prof. Dixon gave a very clear account of some methods of solving (or at least drawing the sting of) common types of integral equations. Dr. Linfoot dealt with a modification of Waring's problem in the theory of numbers; Dr. Wrinch with the use of the recurrence relations of various types of function (especially the Legendre polynomials) to obtain their definite integrals; Prof. Titchmarsh with the discovery of general formulae for all functions having the property of being identical with their own Fourier transforms; and Prof. Berwick with the determination of elliptic functions of arguments of the form $u\sqrt{(-n)}$, given the functions of u. Mr. Chaundy spoke on the hypergeometric equation, and Dr. Bosanquet on the summability of Fourier series. Mr. L. C. Young and Miss R. C. Young dealt in an elementary manner with problems belonging to the conceptual basis of Geometry and Algebra respectively. The application of analysis to numerical problems was not forgotten; Dr. Fisher expounded the somewhat neglected distinction between probability and likelihood, in the course of a paper on inverse probability; the theory of sampling, mechanical and approximate numerical integration, and the construction of logarithm tables were also treated.

But the chief thrills of the meeting were in the physical section. Prof. Daniell's paper on the rate of propagation of flame in inflammable gas was given in the mathematical sub-section; but the sub-section was forsaken by large numbers of mathematicians who crowded to hear Dr. Smith's fascinatingly lucid presidential address on the origin of terrestrial magnetism, and the cause of its variations; the discussion by Prof. Bragg, Prof. Lennard-Jones, and Dr. Bernal, of the various forces which bind atoms together into solid bodies; and Prof. Milne's exposition of his new theory of stellar structure, according to which the main sequence stars have a tiny but enormously dense core, at an immensely higher temperature than has hitherto been suggested, while the all-gaseous state studied by Eddington and supposed by him to be typical of the main sequence appears merely as a limiting case between this and the white dwarfs. The great sensation was, however, undoubtedly provided by Dr. Dirac, who proposed to explain away the apparently solid and massive proton as merely the absence of an electron from a possible state of negative energy; empty space being understood to be any region where there are electrons in all states of negative energy, and none of positive energy. In spite of some quantitative difficulties, Dr. Dirac succeeded in arousing the enthusiasm of a large audience, though many of them seemed to feel that the chief beauty of his theory was its appearance of being utter nonsense.

P. Du Val.

THE SUMMABILITY OF FOURIER SERIES.

By L. S. Bosanquet, D.Phil.*

One of the oldest problems in the theory of Fourier series is that of looking for a criterion that a Fourier series shall converge. No one, however, has been able to find a simple, necessary and sufficient condition for this. Thus, for instance, bounded variation of the function is sufficient but not necessary. Continuity is neither necessary nor sufficient. That is to say, there are functions whose Fourier series converge at points of discontinuity, and others whose Fourier series diverge at points of continuity. If we consider the same problem for Cesàro summability of any particular order, similar difficulties arise.

Certain one-sided results are, however, well known. Fejèr's theorem proves that at a point of continuity of the function the Fourier series is summable by Cesàro's first mean—so that continuity is at any rate sufficient for summability $(C, 1)$. And, on the other hand, it follows from a well-known theorem of Riemann that, at a point of convergence of the Fourier series,

$$\frac{1}{t^2}\int_0^t du \int_0^u \phi(v)dv \to 0,$$

when $t \to 0$, where $\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2s\}$. We may express this by saying that a necessary condition for convergence is that the function shall possess a mean value of the second order at the point x, or that $\phi(t) \to 0$ $(C, 2)$.

Now in 1923 † Hardy and Littlewood reformulated the problem in a rather less ambitious form, and proved that the necessary and sufficient condition that the Fourier series of $f(t)$ be summable by *some Cesàro mean or other*, to sum s, at the point $t=x$, is that there exist an integer κ such that

$$\frac{\kappa}{t^\kappa}\int_0^t (t-u)^{\kappa-1}\phi(u)du \to 0 \text{ when } t \to 0.$$

This expression is obtained by dividing the κth integral of $\phi(u)$ over the range $(0, t)$ by the κth integral of unity over the same range, and we may say that the function has a mean value of order κ at the point x, or that $\phi(t) \to 0$ (C, κ). Thus the necessary and sufficient condition that the Fourier series be summable (C) is that $\phi(t) \to 0$ (C).

More precise results than those originally published by Hardy and Littlewood may be obtained by using non-integral orders. Replacing κ by the positive real number α, and making a trivial substitution, we say generally that $\phi(t) \to 0$ (C, α) if

$$\alpha\int_0^1 (1-v)^{\alpha-1}\phi(tv)dv \to 0,$$

when $t \to 0$, the integral being a Lebesgue integral. We also say that $\phi(t) \to 0$ $(C, 0)$ if it does so in the ordinary sense.

It has now been shown ‡ that

I. If $\phi(t) \to 0$ (C, α), then the Fourier series of $f(t)$ is summable $(C, \alpha+\delta)$ to s for $t=x$, for every $\delta > 0$, when $\alpha \geqq 0$;

II. If the Fourier series of $f(t)$ is summable (C, α) to s for $t=x$, then $\phi(t) \to 0$ $(C, \alpha+1+\delta)$, for every $\delta > 0$, when $\alpha \geqq -1$.

These results extend to general Denjoy-Fourier series (provided in I, $\alpha+\delta \geqq 1$),

* A paper read at the Bristol Meeting of the British Association, September 1930.

† *Math. Zeitschrift*, vol. 19.

‡ See papers by Hardy and Littlewood, Paley, Verblunsky, and the writer in the *Proceedings* of the Cambridge Phil. Soc. and of the London Math. Soc.

and similar results have been obtained for allied series, Fourier integrals, and also power series for continuity up to the circle of convergence.

Consider the condition $\delta > 0$. It is essential to the truth of I and II. In fact it has been shown (though I think only in three cases so far) that the theorems are not true with $\delta = 0$. Thus, to take the case $\alpha = 0$, there are functions which are continuous (*i.e.* $\phi(t) \to 0$ $(C, 0)$) at points at which the Fourier series is not convergent (*i.e.* summable $(C, 0)$). And there are functions whose Fourier series converge at points at which $\phi(t)$ does not tend to zero $(C, 1)$. These results may then be called " best possible " results, in the sense that they are true for every $\delta > 0$, but false for $\delta = 0$. But the question now arises, " Can we not after all improve on these results and obtain something more precise still ? " This problem Dr. Linfoot and I are considering in collaboration.

Now the Cesàro sum of Σa_n of order $\alpha \geqq 0$, or rather the Rieszian mean, which always has the same value if $\alpha \geqq 0$, is

$$\lim_{\omega \to \infty} \sum_{n < \omega} \left(1 - \frac{n}{\omega}\right)^{\alpha} a_n,$$

and knowledge of the existence or non-existence of this limit for certain values of α gives us information about the oscillation of the series Σa_n. Moreover the sum obeys the consistency law that, if it exists for a given α, it also exists, and has the same value, for any $\alpha' > \alpha$.

In order to be able to give more precise information still about the behaviour of Σa_n, we can introduce an additional factor

$$B\left[\log\left\{A \Big/ \left(1 - \frac{n}{\omega}\right)\right\}\right]^{-\beta}$$

and say that the sum (α, β) of Σa_n is

$$\lim_{\omega \to \infty} \sum_{n < \omega} \left(1 - \frac{n}{\omega}\right)^{\alpha} . B\left[\log\left\{A \Big/ \left(1 - \frac{n}{\omega}\right)\right\}\right]^{-\beta} a_n.$$

A and B must be chosen so that $B = (\log A)^{\beta}$, so that each factor tends to unity when $\omega \to \infty$, and the most natural values to choose would be $A = e$ and $B = 1$. However, certain difficulties arise in proving a consistency theorem if we choose these values, and after all there is no reason why they should be chosen rather than any other values. We therefore say (anyway provisionally) that Σa_n is summable (α, β) if the above limit exists and has a unique value *for all sufficiently large values of* A. With this definition we can prove the desired consistency law, namely, that if Σa_n is summable (α, β), then it is also summable (α', β'), and with the same sum, provided either $\alpha' > \alpha$ (and β' has any value) or $\alpha' = \alpha$ and $\beta' > \beta$. Thus we have defined a two-parameter scale of summability, and obviously, by introducing a factor log log ... , we could define a three-parameter scale, and so on indefinitely.

Applying this idea to Fourier series we do obtain more precise results than those previously stated. For instance, to take the case of the theorem that at a point of continuity the Fourier series is summable (C, δ), we find that at such a point the Fourier series is summable $(0, 1 + \delta)$ for every $\delta > 0$ (which says more than that it is summable (C, δ) or $(\delta, 0)$), and moreover a function can be constructed to show that this is no longer true if $\delta = 0$. This shows that the dividing line is not so much between convergence and non-convergence as between summability $(0, 1)$ and non-summability $(0, 1)$. This then, I think, does justify the introduction of this rather clumsy definition, and by introducing further complications we could plainly obtain still better results.

As far as the actual technique is concerned, I will just say (to those who are familiar with the older problem) that what we do is this :

We replace the function

$$\gamma_q(t) = \int_0^1 (1-u)^{q-1} \cos tu\, du$$

by the more complicated function

$$g_{q,\sigma} = \int_0^1 (1-u)^{q-1} . B\,[\log\{A/(1-u)\}]^{-\sigma} \cos tu\, du,$$

and then the formula

$$\text{sum}\,(C, \alpha) = s + \lim_{\omega \to \infty} \frac{2\omega}{\pi} \int_0^\infty \gamma_{1+\alpha}(\omega t)\, \phi(t)\, dt$$

is replaced by the formula

$$\text{sum}\,(\alpha, \beta) = s + \lim_{\omega \to \infty} \frac{2\omega}{\pi} \int_0^\infty g_{1+\alpha,\beta}(\omega t)\, \phi(t)\, dt.$$

We investigate the properties of $g_{q,\sigma}(t)$ and then argue as in the older problem.

L. S. BOSANQUET.

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796. The Master of Trinity College, Dr. Whewell, was unfortunate enough, five and twenty years ago, to fall into one of Nature's traps. . . . In his work on Mechanics, he happened to write *literatim* and *verbatim*, though not lineatim, as follows:

There is no force, however great,
Can stretch a cord, however fine,
Into a horizontal line,
Which is accurately straight.

. . . A little before the occurrence of the preceding, Prof. Woodhouse, in his treatise on Astronomy, was more unfortunate than Mr. Whewell; for he only made the first half of a stanza,—and left the undergraduates to add the second. . . . He was then superintending . . . the completion of the Observatory, which was to be his own official residence; and some dissatisfaction had been expressed at the expense of ornamenting the grounds. So, between them, Woodhouse and the wags made the following:

If a spectator
Be at the equator,
At the point represented by *A*:
So says Mr. Woodhouse,
Who lives in the good house
For which other people must pay.

[There was] an older notice of a stanza, from Smith's "Optics," which has not yet found its other half . . .

If parallel rays
Come contrary ways
And fall upon opposite sides. . . . —

M., *The Athenaeum*, 1846, p. 842 [obviously De Morgan].

797. The celebrated Laplace, then Examiner in the artillery school [at Chalons], was a man of the most serious appearance: his sad and severe face, his black dress, his fringed ruffles, the shade over his eyes—rendered necessary by the state of his sight—gave him a very imposing air. . . . It may be imagined with what anxiety, disquietude, and sinking of the heart we approached the Examiner's table. . . . My brain wandered, and I could not tell M. Laplace what my name was when he asked me! [Laplace was considerate, and the candidate, aged 18, became a sub-lieutenant of artillery.]—Quoted from *Mémoires du Duc de Raguse* [Marshal Marmont], 1792-1832, *Athenaeum*, 1856, p. 1399.

AN APPROXIMATE CONSTRUCTION FOR THE DIVISION OF AN ANGLE.

By Prof. D. M. Y. Sommerville, D.Sc.

1. The following simple approximate construction for trisecting an angle is due to Mr. Ralph H. Jones of Auckland, N.Z.

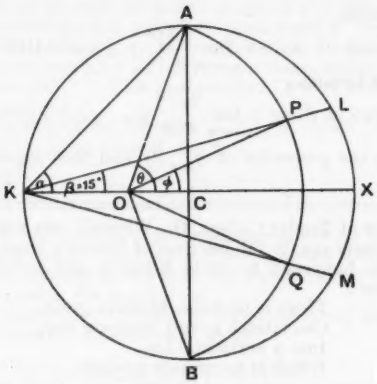

Fig. 1.

Let BOA be the given angle (Fig. 1). Make $OA=OB$, join AB, and draw $OX \perp AB$ cutting AB in C. On AB as diameter describe a circle cutting XO, produced if necessary, in K. With centres A and B and radius $=AC$ draw arcs to cut AXB in L and M respectively, thus trisecting the semicircular arc AXB. Join KL, KM, cutting the circle, centre O and radius OA, in P and Q. OP and OQ are the approximate trisectors.

2. To find the error of this approximation, let the given angle $=2\theta$, so that $XOA=\theta$, and let $XOP=\phi$. The angle $OKP=15°$ and $OPK=\phi-15°$, $OKA=45°$ and $OAK=\theta-45°$. Then since $OP=OA$,

$$\frac{\sin(\phi-15°)}{\sin 15°}=\frac{\sin(\theta-45°)}{\sin 45°}. \quad \text{.....(2.1)}$$

This equation is evidently satisfied identically in the three cases:

(i) $\theta=0$, $\phi=0$; (ii) $\theta=45°$, $\phi=15°$; (iii) $\theta=90°$, $\phi=30°$,

so that for $2\theta=0$, 90° and 180° the construction is exact.

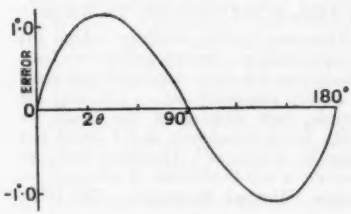

Fig. 2.

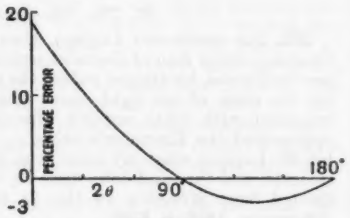

Fig. 3.

The accompanying graph (Fig. 2) gives the error $\frac{1}{3}\theta - 2\phi$ in the angle POQ for values of 2θ from 0 to 180°. Fig. 3 gives the percentage error. It is of interest to note that, while the actual error is zero when the angle is zero, the percentage error tends to a finite limit. As $\theta \to 0$ we have

$$\text{Lim } \phi/\theta = \tan 15° = 0{\cdot}268,$$

and the limit of the percentage error is

$$100(1 - 3\tan 15°) = 100(3\sqrt{3} - 5) = 19{\cdot}6.$$

3. The construction can be generalised as follows (Fig. 1), considering $\theta = XOA$ as the angle to be divided. Draw a circle with centre O and any radius OA. Choose two arbitrary angles α and β. Make $XKA = \alpha$ and draw KP making angle $XKP = \beta$ and cutting the circle in P. Join OP. Let $XOP = \phi$. Then

$$\frac{\sin(\phi - \beta)}{\sin\beta} = \frac{\sin(\theta - \alpha)}{\sin\alpha}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.1)$$

This fits at $\theta = 0$, and by choosing α and β it can be made to fit for two other values of θ. Mr. Jones' construction, which fits at $\theta = 45°$ and 90° determines the angles $\alpha = 45°$ and $\beta = 15°$. If $\alpha = 3\beta$, it fits at $\theta = 0$, α and 2α.

4. More generally, the same construction provides an approximate construction for θ/n when θ is given. Thus, taking $\alpha = n\beta$, the equation

$$\frac{\sin(\phi - \beta)}{\sin\beta} = \frac{\sin(\theta - n\beta)}{\sin n\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.1)$$

is satisfied by $\theta = 0$, $\phi = 0$; $\theta = n\beta$, $\phi = \beta$; $\theta = 2n\beta$, $\phi = 2\beta$.

The error $\theta/n - \phi = u$ has values equal and opposite in sign for values of θ equal to $n\beta \pm t$. To find the maximum error between O and $n\beta$ or between $n\beta$ and $2n\beta$ we have $\frac{du}{d\theta} = 0$ while $\phi = \theta/n - u$. Hence

$$\frac{\cos(\phi - \beta)}{n\sin\beta} = \frac{\cos(\theta - n\beta}{\sin n\beta}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.2)$$

Eliminating ϕ between (4.1) and (4.2) we have

$$\sin^2(\theta - n\beta) + n^2\cos^2(\theta - n\beta) = \sin^2 n\beta/\sin^2\beta,$$

whence
$$(n^2 - 1)\cos 2(\theta - n\beta) = 2\sin^2 n\beta/\sin^2\beta - n^2 - 1. \quad \ldots\ldots\ldots (4.3)$$

Taking $n\beta = 45°$ the value of θ which gives the maximum error is given by

$$(n^2 - 1)\sin 2\theta = \text{cosec}^2\beta - n^2 - 1, \quad \ldots\ldots\ldots\ldots\ldots (4.4)$$

and, for all integral values of $n \geqslant 3$, an approximate value of θ is given by

$$\sin 2\theta = \frac{16}{\pi^2} - 1,$$

whence
$$\theta = 19{\cdot}20°. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.5)$$

Then from (4.1)

$$\sin\left(\frac{45°}{n} - \phi\right) = \sqrt{2}\sin 25{\cdot}8° \sin\frac{45°}{n}$$

$$= 0{\cdot}6155 \sin\frac{45°}{n}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.6)$$

The maximum error is then as follows:

n	max. error.
3	0·56°
5	0·36°
7	0·26°
9	0·22°

This of course tends to zero as $n \to \infty$.

The relative error, *i.e.* the ratio of u to θ/n, $=1-n\phi/\theta$. As $\theta\to 0$ and $\phi\to 0$ we have from (4.1)

$$\phi\cot\beta=\theta\cot n\beta,$$

and therefore the relative error tends to the limit

$$1-n\cot n\beta\tan\beta.$$

With $n\beta=45°$, as $n\to\infty$ the relative error at $\theta=0$ tends to the value

$$1-\tfrac{1}{4}\pi=0{\cdot}2146,$$

and the limiting value of the percentage error as $\theta\to 0$ is 21·5, not much greater than that for $n=3$, viz. 19·6.

The relative error is zero at $\theta=\alpha$ and at $\theta=2\alpha$, and between these values of θ it is negative and has a numerical maximum, which will occur a little before the numerical maximum of the actual error which is at $\theta=90°-19{\cdot}20°$ $=70{\cdot}80°$. In the graph (Fig. 4) of the actual error, $u=\theta/n-\phi$, the relative

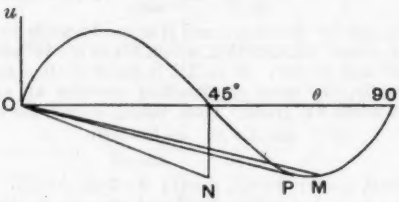

FIG. 4.

error is proportional to the gradient u/θ of the radius vector drawn from O. Hence the maximum relative error corresponds to the gradient of the tangent OP. If M is the lowest point on the graph and N the point at the same level below $\theta=45°$, OP lies between OM and ON.

Still assuming $n\beta=45°$, and substituting $\theta=70{\cdot}80°$ in (4.1), we get

$$\sin\left(\phi-\frac{45°}{n}\right)=0{\cdot}6155\sin\frac{45°}{n}.$$

When n is large, this gives approximately

$$\phi-\frac{\pi}{4n}=0{\cdot}6155\,\frac{\pi}{4n},$$

i.e.

$$\phi=\frac{1{\cdot}6155\pi}{4n},$$

and the relative error lies between

$$\frac{1{\cdot}6155\pi}{4\theta}-1 \quad\text{and}\quad \left(\frac{1{\cdot}6155\pi}{4\theta}-1\right)\frac{\theta}{\alpha},$$

where $\theta=70{\cdot}80°$ and $\alpha=45°$.

On calculation we find the values 0·0268 and 0·0422. Hence the maximum percentage error in the range between $\theta=45°$ and $\theta=90°$ tends to lie between 2·7 and 4·2. For $n=3$ it is 2·45 at $\theta=67°$ (approx.); for $n=10$ it is about 2·7 at $\theta=70°$ (approx.).

D. M. Y. SOMMERVILLE.

Victoria University College, Wellington, N.Z.

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798. [Kirke White's] life will affect you, for he fairly died of intense application. Cambridge finished him. When his nerves were already so overstrained that his nights were utter misery, they gave him medicines to enable him to hold out during examination for a prize! The horse won—but he died after the race.—Southey to Duppa, quoted *Athenaeum*, 1850, p. 260.

MATHEMATICAL NOTES.

983. [D. 5. 1.] *An elementary proof of Saalschütz's theorem on generalised hypergeometric series.*

As long ago as 1890, Saalschütz * gave the sum of a particular terminating hypergeometric series of the type

$$ {}_3F_2\left[\begin{matrix} a, & \beta, & -n; \\ \delta, & \epsilon & \end{matrix}\; 1\right] \equiv \sum_{r=0}^{n} \frac{(a)_r(\beta)_r(-n)_r}{r!\,(\delta)_r(\epsilon)_r}, $$

where $(a)_r = a(a+1)(a+2)\ldots(a+r-1)$, and n is a positive integer.

The theorem is that, if

$$ \delta+\epsilon = a+\beta-n+1, $$

that is, if the sum of the denominator parameters exceeds the sum of the numerator parameters by unity, then

$$ {}_3F_2\left[\begin{matrix} a, & \beta, & -n; \\ \delta, & \epsilon & \end{matrix}\; 1\right] = \frac{(\delta-a)_n(\delta-\beta)_n}{(\delta)_n(\epsilon)_n}. $$

Much use of this theorem has recently been made to obtain much more general results concerning generalised hypergeometric series.† Several proofs of the theorem have been given. For example, an algebraic proof has been given by Dougall,‡ who also showed that the relation

$$ F(a,\ \beta;\ \gamma;\ x) = (1-x)^{\gamma-a-\beta}F(\gamma-a,\ \gamma-\beta;\ \gamma;\ x), $$

where F is the ordinary hypergeometric function, can be deduced from Saalschütz's formula by comparing coefficients of x^n on the two sides. On the other hand, this last identity can conversely be used to prove Saalschütz's formula.§

I find that a very elementary proof of the theorem can be given using nothing more advanced than gamma functions and Vandermonde's theorem which can be written in the form

$$ F(a,\ -n;\ b;\ 1) = \frac{(b-a)_n}{(b)_n}. $$

We first prove the identity

$$ (1) \qquad \sum_{r=0}^{n} \frac{(a)_r(-n)_r}{r!\,(\delta)_r}(1-x)^r = \frac{(\delta-a)_n}{(\delta)_n} \sum_{r=0}^{n} \frac{(a)_r(-n)_r x^r}{r!\,(1+a-\delta-n)_r}. $$

This follows by noting that the coefficient of x^r on the left is

$$ \sum_{m=r}^{n} \frac{(a)_m(-n)_m}{m!\,(\delta)_m} \frac{(-1)^r m!}{(m-r)!\,r!}, $$

and putting $m = r+t$, this becomes

$$ \frac{(-1)^r(a)_r(-n)_r}{r!\,(\delta)_r} \sum_{t=0}^{n-r} \frac{(a+r)_t(-n+r)_t}{t!\,(\delta+r)_t} = \frac{(-1)^r(a)_r(-n)_r}{r!\,(\delta)_r} F(a+r,\ -n+r;\ \delta+r;\ 1). $$

Using Vandermonde's theorem to sum the series on the right, we obtain (1).

Now multiply (1) by $x^{\epsilon-\beta-1}(1-x)^{\beta-1}$ and integrate from 0 to 1, and we get

$$ \sum_{r=0}^{n} \frac{(a)_r(-n)_r}{r!\,(\delta)_r} \frac{\Gamma(\epsilon-\beta)\Gamma(r+\beta)}{\Gamma(\epsilon+r)} = \frac{(\delta-a)_n}{(\delta)_n} \sum_{r=0}^{n} \frac{(a)_r(-n)_r}{r!\,(1+a-\delta-n)_r} \frac{\Gamma(\beta)\Gamma(\epsilon-\beta+r)}{\Gamma(\epsilon+r)}, $$

* L. Saalschütz, "Eine Summationsformel," *Zeitschrift für Math. u. Phy.* 35 (1890), 186-188.

† See, for example, W. N. Bailey, "Some identities involving generalised hypergeometric series," *Proc. London Math. Soc.* (2), 29 (1929), 503-516.

‡ J. Dougall, "On Vandermonde's theorem, and some more general expansions," *Proc. Edinburgh Math. Soc.* 25 (1907), 114-132.

§ Cf. G. H. Hardy, "A chapter from Ramanujan's note-book," *Proc. Camb. Phil. Soc.* 21 (1923), 492-503.

which can be written in the form

$$(2) \qquad {}_3F_2\left[\begin{matrix}a, \beta, -n; \\ \delta, \epsilon\end{matrix}\ 1\right] = \frac{(\delta-a)_n}{(\delta)_n}\,{}_3F_2\left[\begin{matrix}a, \epsilon-\beta, -n; \\ 1+a-\delta-n, \epsilon\end{matrix}\ 1\right].$$

Now if $\delta+\epsilon=a+\beta-n+1$, the series ${}_3F_2$ on the right of (2) reduces to

$$F(a, -n; \epsilon; 1),$$

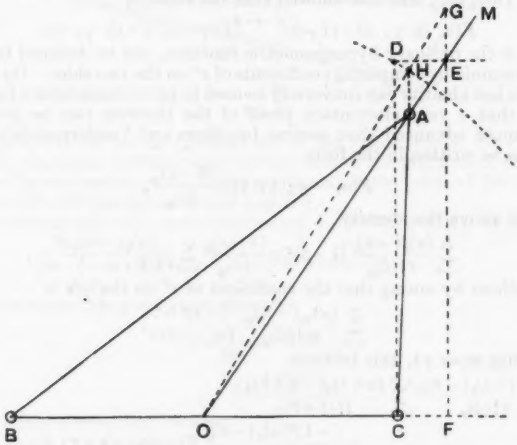
which can be summed by Vandermonde's theorem, and we are led to Saalschütz's formula. W. N. BAILEY.

984. [K¹. 20. e.] *Given the base BC of a triangle ABC, and the direction OM of the median which bisects BC in O, and also given the sum of the sides, to find the vertex A.*

Let $BC=2c$, let the given median make angle α with OC, and let $2a$ be the sum of the sides AB, AC. Then, if $b^2=a^2-c^2$, the vertex A lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and on the line $y=x\tan\alpha$.

$$\therefore\ x^2(b^2+a^2\tan^2\alpha)=a^2b^2.$$

Now, let $a\tan\alpha=b\tan\theta$, and we get $x=a\cos\theta$.



Hence the construction:—

With centre O, radius equal to half the sum of AB, AC, describe a circle cutting in D the ordinate CD drawn perpendicular to BC.

Draw DE parallel to BC to cut OM in E, draw the ordinate EF and on it make $FG=\frac{1}{2}(AB+AC)$.

Join OG, cutting the circle in H; then the ordinate of H cuts OM in the required vertex A. ALFRED LODGE.

985. [C. 2. a.] *On the Method of Integration by Substitution.*

If $x=\phi(y)$ and both $f(x)$ and $\phi(y)$ are "reasonably behaved" functions of x and y respectively, then it is known that

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\phi(y))\frac{d\phi(y)}{dy}dy\,d$$

where $a=\phi(\alpha)$ and $b=\phi(\beta)$.

After proving this result in my lectures I frequently set the following question:

"Let

(1) $$(x-a)(x-b)=y,$$

then

(2) $$\int_a^b f(x)dx=\int_0^0 \psi(y)\frac{dx}{dy}dy=0,$$

where $\psi(y)$ is the result of substituting in $f(x)$ for x in terms of y. Criticise the above reasoning and show how to evaluate the left-hand integral of (2) correctly by means of the substitution (1)."

To discover the fallacy in this reasoning draw the parabola given by equation (1) and let it cut the x axis in the points A and B so that $OA=a$, $OB=b\ (>a)$ where O is the origin (Fig. 1). Let C be the lowest point of the

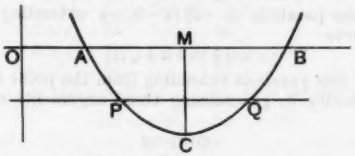


FIG. 1.

parabola and M the foot of the ordinate through C (evidently as $a<x<b$, $(x-a)(x-b)$ is negative and so the arc ACB lies wholly below the x axis). It is easily shown that

$$OM=\tfrac{1}{2}(b+a) \quad \text{and} \quad MC=-\tfrac{1}{4}(b-a)^2.$$

Let any line parallel to the x axis and distant y from it cut the parabola in the points P and Q, one of which, P say, must be to the left and the other to the right of C. Then on solving (1) for x in terms of y the abscissae of P and Q are given respectively by

$$\tfrac{1}{2}(b+a-\sqrt{\Delta}) \quad \text{and} \quad \tfrac{1}{2}(b+a+\sqrt{\Delta}),$$

where* $$\Delta=(b-a)^2+4y.$$

Evidently then, as x varies from O to $\frac{1}{2}(a+b)$ (1) gives rise to the substitution:

(3) $$x=\tfrac{1}{2}(b+a-\sqrt{\Delta}) \quad \text{with} \quad dx/dy=-\Delta^{-\frac{1}{2}},$$

whilst as x varies from $\frac{1}{2}(a+b)$ to b (1) gives rise to the substitution:

(4) $$x=\tfrac{1}{2}(b+a+\sqrt{\Delta}) \quad \text{with} \quad dx/dy=\Delta^{-\frac{1}{2}}.$$

Hence
$$\begin{aligned}\int_a^b f(x)dx &= -\int_0^{-\frac{1}{4}(b-a)^2} f\{\tfrac{1}{2}(b+a-\sqrt{\Delta})\}\,dy/\sqrt{\Delta} \\ &\quad +\int_{-\frac{1}{4}(b-a)^2}^0 f\{\tfrac{1}{2}(b+a+\sqrt{\Delta})\}\,dy/\sqrt{\Delta} \\ &= -\int_0^{-\frac{1}{4}(b-a)^2} [f\{\tfrac{1}{2}(b+a-\sqrt{\Delta})\}+f\{\tfrac{1}{2}(b+a+\sqrt{\Delta})\}]\,dy/\Delta^{-\frac{1}{2}}.\end{aligned}$$

An illustration is afforded by

$$\begin{aligned}\int_a^b dx &= -\int_0^{-\frac{1}{4}(b-a)^2} 2\,dy/\Delta^{-\frac{1}{2}} \\ &= -\Delta^{\frac{1}{2}}\int_0^{-\frac{1}{4}(b-a)^2} dy \\ &= b-a.\end{aligned}$$

* $\sqrt{(b-a)^2+4y}$ denotes the positive square root of $(b-a)^2+4y$ as y varies from 0 to $-\frac{1}{4}(b-a)^2$ and then back to 0. It is necessary to write the square root as $\sqrt{(b-a)^2+4y}$ and not as $\sqrt{(a-b)^2+4y}$, since when $y=0$ the first form gives us $b-a$ which is positive, as it should be, while the second form gives us $a-b$ which is negative.

In general the substitution (1) will be made for the integral $\int_c^d f(x)dx$ where c and d are not necessarily the same as a and b. It is evident from the above reasoning that if

(i) $c < d \leqq \frac{1}{2}(a+b)$, the substitution (3) only is used throughout the range (c, d);

(ii) $c < \frac{a+b}{2}$, $d > \frac{1}{2}(a+b)$, the substitution (3) is used in the range $x=c$ to $x=\frac{1}{2}(a+b)$ and the substitution (4) is used in the range $x=\frac{1}{2}(a+b)$ to $x=d$;

(iii) $d > c > \frac{1}{2}(a+b)$, substitution (4) is used throughout the range (c, d).

It is worth while observing that the curve

$$x=\tfrac{1}{2}(b+a-\sqrt{\Delta})$$

is the left half of the parabola $(x-a)(x-b)=y$ extending up to the point C (Fig. 1) and the curve

$$x=\tfrac{1}{2}(b+a+\sqrt{\Delta})$$

is the right half of this parabola extending from the point C.

There is no difficulty in generalising these arguments so as to cover the substitution

$$\phi(x)=y \tag{5}$$

where $\phi(x)$ is a polynomial in x of degree n. The curve (5) will have n roots all functions of y, say $x_1(y)$, $x_2(y)$, ... $x_n(y)$, and will have $n-1$ turning points. Let M_r the foot of the ordinate of the rth turning point, then for all values of x between M_{r-1} and M_r* there corresponds one of the roots of (5), $x_r(y)$ say. Then on making the substitution (5) in

$$\int_a^b f(x)dx,$$

for that part of (a, b) which lies inside (M_r, M_{r+1}) we must use the substitution $x=x_r(y)$ together with $\frac{dx}{dy}=\frac{dx_r(y)}{dy}$.

C. Fox.

986. [S. 1. a.] *Particle equivalents for the resultant thrust on a plane quadrilateral completely immersed in a liquid of uniform density.*

Let $A_1A_2A_3A_4$ be the quadrilateral, and O be the intersection of its diagonals. Let the depths of the vertices and of O be a_1, a_2, a_3, a_4 and a_0. Let the area of the quadrilateral be Δ and of the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ be Δ_1, Δ_2, Δ_3, Δ_4.

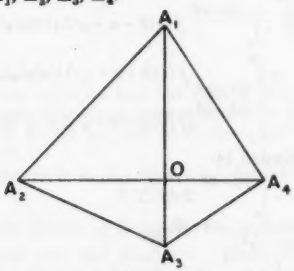


FIG. 1.

Let the depth of the C.G. of the quadrilateral be h, and of the triangles be h_1, h_2, h_3, h_4.

* We may consider M_0 as $-\infty$ and M_n as $+\infty$.

Then
$$3h_1 = a_2 + a_3 + a_4, \text{ etc.} \quad \ldots\ldots(1)$$
$$3h = a_1 + a_2 + a_3 + a_4 - a_0 \quad \ldots\ldots(2)$$
$$\Delta = \Delta_1 + \Delta_3 = \Delta_2 + \Delta_4 \quad \ldots\ldots(3)$$
$$\Delta h = \Delta_1 h_1 + \Delta_3 h_3 = \Delta_2 h_2 + \Delta_4 h_4 \quad \ldots\ldots(4)$$

Taking the density of the liquid as one-twelfth, the thrust on the triangle $A_1A_2A_3$, say, is known to be the resultant of parallel forces at A_1, A_2 and A_3, equal to $\Delta_4(2a_1+a_2+a_3)$, $\Delta_4(a_1+2a_2+a_3)$, $\Delta_4(a_1+a_2+2a_3)$
or $\Delta_4(a_1+3h_4)$, $\Delta_4(a_2+3h_4)$, $\Delta_4(a_3+3h_4)$.

Hence dividing the quadrilateral first by the diagonal A_1A_3, and then by A_2A_4, the sum of the parallel forces at A_1 which contribute to double the total thrust is
$$\Delta_4(a_1+3h_4)+\Delta_2(a_1+3h_2)+\Delta_3(a_1+3h_3).$$

This
$$\begin{aligned}
&= a_1(\Delta_2+\Delta_3+\Delta_4)+3(\Delta_2h_2+\Delta_3h_3+\Delta_4h_4)\\
&= a_1(2\Delta-\Delta_1)+3(2\Delta h-\Delta_1h_1) && \text{by (3) and (4)}\\
&= 2\Delta(a_1+3h)-\Delta_1(a_1+3h_1)\\
&= 2\Delta(a_1+3h)-\Delta_1(a_1+a_2+a_3+a_4) && \text{by (1)}\\
&= 2\Delta(a_1+3h)-\Delta_1(a_0+3h). && \text{by (2)}
\end{aligned}$$

The contribution of the parallel forces at A_3 is similarly
$$2\Delta(a_3+3h)-\Delta_3(a_0+3h).$$

The resultant of $-\Delta_1(a_0+3h)$ at A_1 and $-\Delta_3(a_0+3h)$ at A_3 is
$-(\Delta_1+\Delta_3)(a_0+3h)$ or $-\Delta(a_0+3h)$ at O,
since O divides A_1A_3 in the proportion $\Delta_3 : \Delta_1$.

Again, the contributions of the parallel forces at A_2 and A_4 reduce to $2\Delta(a_2+3h)$ at A_2, $2\Delta(a_4+3h)$ at A_4 and $-\Delta(a_0+3h)$ at O.

Hence altogether the thrust on the quadrilateral is the resultant of parallel forces at A_1, A_2, A_3, A_4 and O proportional to (a_1+3h), (a_2+3h), (a_3+3h), (a_4+3h) and $-(a_0+3h)$ respectively.

Cor. 1. The coordinates of the centre of pressure of the quadrilateral can thus be found, and in particular the depth of the C.P. is
$$\begin{aligned}
&\frac{a_1(a_1+3h)+a_2(a_2+3h)+a_3(a_3+3h)+a_4(a_4+3h)-a_0(a_0+3h)}{(a_1+3h)+(a_2+3h)+(a_3+3h)+(a_4+3h)-(a_0+3h)}\\
&= [\Sigma a^2+3h\Sigma a-(a_0+3h-3h)(a_0+3h)]/(12h)\\
&= [\Sigma a^2+3h\Sigma a-(\Sigma a)^2+3h\Sigma a]/(12h), && \text{by (2)}\\
&= \Sigma a/2-\Sigma a_r a_s/(6h).
\end{aligned}$$

Cor. 2. For a parallelogram the equivalent parallel forces are proportional

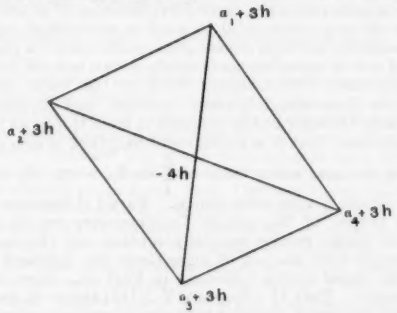


FIG. 2.

to a_1+3h, a_2+3h, a_3+3h, a_4+3h and $-4h$, or to a_1+2h, a_2+2h, a_3+2h and a_4+2h at the respective vertices. N. M. GIBBINS.

REVIEWS.

Advanced Trigonometry. By C. V. DURELL and A. ROBSON. Pp. vi + 336. 8s. 6d. 1930. (Bell.)

Trigonometry. By A. W. SIDDONS and R. T. HUGHES. Pp. vi + 416. Part I, 1s. 9d. Part II, 2s. 6d. Part III, 1s. 9d. Part IV, 3s. 6d. Parts I-II, 3s. 6d. Parts I-III, 4s. 6d. Parts III-IV, 4s. 6d. Parts I-IV, 7s. 6d. 1928-29. (Camb. Univ. Press.)

Plane Trigonometry. By H. S. CARSLAW. Pp. viii + 342. Part I, 3s. Part II, 3s. Parts I-II, 5s. 1930. (Macmillan.)

(1) Teachers of mathematical specialists in schools have long been looking for text-books which shall fulfil two conditions. Firstly, they must contain plenty of exercises for the pupil; and, secondly, the bookwork must be set out on rigorous lines in a manner which is suited to the tender years of the budding mathematician. Messrs. Durell and Robson's book on Advanced Trigonometry has fourteen chapters; each chapter includes sets of exercises in the body of the text and ends with a miscellaneous set which, more often than not, is divided into two parts, containing easy and harder questions. So the first condition is amply satisfied. And, in the matter of bookwork, the authors have, in the writer's opinion, just about hit the mark.

A knowledge of the earlier parts of Trigonometry is naturally taken for granted, and the reader is also assumed to be acquainted with Algebra up to and including the Binomial Theorem and with the elements of the Calculus, including integration by parts. This being understood, the treatment may be said to be complete in itself. There is a refreshing absence of reference to standard works; the theory of the logarithmic and exponential functions of a real variable is included; a good chapter on complex numbers precedes the discussion of De Moivre's Theorem and all that follows therefrom; the thorny subject of convergence does not crop up too often, and when it does the treatment is *ab initio*.

In briefest outline, the work is in three sections. Chapters I-III are mainly a revision course (properties of the triangle, equations, inverse functions, etc.); Chapters IV-VII are concerned with the real variable, and Chapters VIII-XIII with the complex variable. Infinite products are not treated.

It would take too long to mention in detail the many excellent features of this book. Here are a few, written down almost at random. The properties of $\log x$, e^x are developed from the definition of $\log x$ as a definite integral; the series for $\sin x$, $\cos x$, $\log(1+x)$, e^x are obtained by using the methods of the Calculus and paying attention to "limits" and "remainders"; the chapters on the complex variable contain a careful explanation of principal values; the expression of $\cos n\theta$ as a polynomial in $\cos\theta$ is established by differentiating and obtaining a relation between consecutive coefficients—a preferable method to the more usual one of equating coefficients in two infinite series.

The book really ends with Chapter XIII on the many valued functions $\operatorname{Log} w$, z^w and the generalised inverse circular and hyperbolic functions Chapter XIV merely affords further practice in such things as identities, elimination and inequalities; but it is worth recording that it also contains a satisfactory proof that the maximum value of $\sum_1^5 \sin\theta_r$, when $\Sigma\theta_r = 5\alpha$, is $5\sin\alpha$.

(2) This book covers a very wide range. Part I (Chapters I-IV) takes the pupil through a course of Numerical Trigonometry up to the solution of triangles. Obtuse angles receive special attention and the formulae required for solving a triangle with the aid of logarithms are obtained by geometrical methods, the more usual proofs following in Part II. Exercises in three dimensions occur early. Part II (Chapters V-XIII) treats of the general angle, identities, addition theorems, circular measure, equations, elimination and the properties of the triangle and quadrilateral. There is a good section on the graphical solution of equations, and many examples occur which relate to other subjects such as Mechanics and Physics. Part III (Chapters XIV-XVI) deals

with complex numbers, De Moivre's Theorem, the exponential forms of the circular functions, the hyperbolic functions and the summation of finite series; and Part IV (Chapters XVII-XXIV) with the circular, exponential and logarithmic functions of a complex variable, expansions in finite and infinite series, factorisation, infinite products, and symmetrical functions of the roots of equations.

A large number of examples are worked out in the text and there are abundant exercises. In particular, there are 11 revision papers in Part I, and 12 in Part II; and there is a set of 120 miscellaneous questions in the first Part, 75 in the second, 118 in the third and 138 in the fourth. Such profusion will gladden the hearts of teachers who use this book; they will have the pleasant task of picking and choosing rather than the irritating one of searching elsewhere for more.

As to subject matter, Part I provides just the sort of course that a beginner needs, while the pupil who has mastered Part II will be thoroughly prepared for starting Advanced Trigonometry. The remainder of the book is intended for the specialist. The treatment is, more or less, on the lines of the older school text-books; but, where they omit to point out an implied assumption or to justify a line of argument which needs it, the authors have set themselves the task of repairing the omission. Certain portions of the text require amplifying, and when this has been done Parts III and IV will form a valuable aid to the teaching of Higher Trigonometry and will be welcomed especially by those who wish to stick close to the traditional mode of presenting the subject.

(3) Prof. Carslaw's book first appeared in 1900. As must now be well known, Part I deals with Elementary Trigonometry up to the solution of triangles, concluding with a chapter on circular measure, while Part II includes the geometrical properties of triangles and their associated circles, De Moivre's Theorem and its applications, trigonometrical equations and finite series. The text is written with the care and precision that one naturally expects from the author, but the rapacious schoolmaster would appreciate a larger supply of exercises to be worked.

The last two chapters form the chief feature which distinguishes this third edition from previous editions. These give proofs, based upon Tannery's Theorem, of the infinite series and products for $\sin x$ and $\cos x$. Such work may suit the needs of college students, but most schoolmasters will regard it as undesirable pabulum for their mathematical specialists and will deplore the paucity of exercises in these final chapters: there is one exercise on infinite products. C. J. A. T.

Plane Trigonometry. By H. S. CARSLAW. Third edition, revised. Pp. xviii + 330 + xii. 5s. 1930. (Macmillan.)

Advanced Trigonometry. By C. V. DURELL and A. ROBSON. Pp. viii + 336. 8s. 6d. 1930. (Bell.)

[*These notices were written in fulfilment of a promise to the late Editor.*]

(1) In reading this book critically we have to bear in mind the sub-title: An elementary text-book for the higher classes of secondary schools and for colleges. The early chapters are clearly written, but they are in some respects disappointing, perhaps because they have been altered very little since they appeared twenty years ago. Afraid to rely from the beginning on the definitions of cosine and sine as projections, Prof. Carslaw has to discuss the ratios of 0° and 90° by means of vanishing triangles: "When the angle is very small, and grows still smaller, MP is small and grows still smaller, so that with the vanishing of the angle, the sine of the angle also vanishes". This is not reassuring—from the man who knows more about Gibbs' phenomenon than anyone else alive! Again, in dealing with the ratios of $90° - \theta$, $90° + \theta$, and $180° - \theta$, Prof. Carslaw draws figures for every quadrant, and his proof of identities is only that "whatever be the size of the angle θ, it is clear from the figures" that the necessary congruences of triangles and relations of signs persist. From a preliminary remark that "the proofs which are given hold word for word and letter for letter, for any possible figure", one might suppose

that the ultimate reliance was on general principles and that the figures were multiplied only by way of confirmation for the sake of the less ready students, but this generous assumption is belied by the next sentence, which definitely throws the burden of proof on the several figures, leaving the learner to wonder, presumably, whether the existence of comprehensive formulae is more than a series of happy accidents.

This amounts to saying that at first Prof. Carslaw is concerned to be lucid to beginners, rather than to introduce to serious students the basis of his subject in a form that can be regarded later as satisfactory. But what of the developments? Readers of the *Gazette* are familiar with Prof. Carslaw's views on rigour, and will look with interest to see how far he is able to put them into practice. Unfortunately, this is precisely what they will not discover, for the book is composed on the understanding that a course of analysis is being followed concurrently. Complex numbers are brought in without explanation as soon as they are wanted, and "the reader is supposed familiar with the arithmetical definition of the limit of $\phi(n)$ when $n \to \infty$, where $\phi(n)$ is a function of n, defined for all positive integers"; this is explicit, and implicitly equal familiarity is supposed with the nature of a limit of a function of a continuous variable and with the essential property of a continuous function. The author makes comments on the fundamental ideas of analysis, without shouldering any responsibility.

In one respect the comments on analytical ideas are misleading. Again and again the language seems to suggest that in an approach to a limit of a sequence each term is necessarily nearer to the limit than its predecessor; it is not to be disputed that in the cases described the approach is of this steady kind, but this is, if anything, the more reason for an emphatic warning that there is nothing in the nature of a limit to preclude an erratic approach. Apart from this flaw, Prof. Carslaw's comments are as illuminating as we should expect, and teachers of analysis will find them useful; space forbids quotation, and in criticism it need be said only that while there are few slips, the use of series in the evaluation of limits has an eighteenth century recklessness.

Towards the end, Prof. Carslaw outstrips the general analysis on which he has been relying, and in a series of sections marked for more advanced students he deals with Tannery's theorem and its applications to series and products, along lines reproduced in the *Gazette* last May. There are, of course, no technical mistakes in this work, and perhaps the dislike, which I share with my contemporaries, of investigating products always as logarithmic series, is a groundless prejudice. Every teacher must decide for himself whether his class can appreciate the details of the arguments in these concluding sections, or whether results are to be given deliberately without proof. The doctrine to which we all subscribe now has never been set forth better than by Tannery himself in the preface to his *Leçons d'Algèbre et d'Analyse*, 1906: "Toutefois, lorsqu'il m'est arrivé de laisser de côté certains raisonnements, indispensables pour établir un théorème en toute rigueur logique, j'ai cru devoir en avertir le lecteur et lui signaler les difficultés ou les lacunes. J'ai horreur d'un enseignement qui n'est pas toujours sincère: le respect de la vérité est la première leçon morale, sinon la seule, qu'on puisse tirer de l'étude des sciences".

It remains to be said that intrinsically Prof. Carslaw's book deals with the trigonometry only of the real variable. De Moivre's theorem is used, the exponential equivalents of the sine and cosine are given, and the hyperbolic functions are related formally to circular functions of a purely imaginary variable, but there is no treatment of the functions of a mixed argument. The value of the drill involved in questions on these functions may be small, but boys undoubtedly enjoy it, and I cannot help feeling that accounts of some functional relations between two complex variables and of the distribution of values of some simple singly-periodic functions of one complex variable are better suited than Tannery's theorem, even to the boy about to pursue mathematics at the university, as well as being far more likely to interest the non-specialist and to give him some inkling of what he is missing.

Prof. Carslaw has given ample evidence elsewhere that he can write an excellent text-book; to me this trigonometry, in which he is doubly hampered

by an outworn tradition and an old framework, seems thorough but not stimulating.

(2) If there is one point of view which can be said to differentiate advanced trigonometry from elementary in a field common to both, it is that from which we recognise the generality of formulae; the first two chapters of Messrs. Durell and Robson's book, which are concerned with formulae in the geometry of triangles and quadrilaterals, are out of place, because their admirable thoroughness takes the form of pointing out where an assumption that an angle is acute has been made, not of explaining how it is that with one or two intelligible exceptions formulae can be demonstrated in such a way that the question whether a triangle is acute-angled or not need not be asked.

But when we come to the analytical trigonometry, which is the main subject of the book, disappointment ceases. This is not the old familiar track, with just a bridge strengthened here, a pitfall avoided there, and notice-boards everywhere: the ground has been surveyed afresh, and an original and inspiring course has been planned. A serious difficulty is inherent in the subject: analytical trigonometry is a part of analysis, and any account of it which is not incorporated into a general course must avoid the extremes of devoting an excessive amount of attention to general principles and of taking for granted that the concurrent course of analysis is satisfactory. Messrs. Durell and Robson promise a companion volume, but they have inserted here so much as is necessary to ensure that teachers using their book will know what kind of setting to construct. Indeed, they have done more: in the brilliant chapter on the nature of complex numbers, which surely belongs to the other volume, they have given a model from which the construction of the setting may be attempted.

Differentiation and integration are used freely throughout. This is probably where the conscientious teacher brought up in a school of algebraical rigour will feel least confidence in constructing the setting himself. Has the appeal to plausibility been banished scrupulously from the discussion of $\sqrt{2}$ and e only to return under the guise of inequalities relating to definite integrals? Not if the basis of the calculus is sufficiently exact, but we are impatient to be told what basis the authors think satisfactory for school purposes.

By a strange oversight, a rigorous determination of the sums of the series

$$\cos a - \tfrac{1}{2}\cos 2a + \tfrac{1}{3}\cos 3a - \dots, \quad \sin a - \tfrac{1}{2}\sin 2a + \tfrac{1}{3}\sin 3a - \dots,$$

is said to be beyond the scope of the book. An inequality governing the integral of a remainder term is used repeatedly in evaluating a series, and all that is necessary in this case to secure an appropriate inequality after integrating the sums to n terms of the elementary series

$$\sin a - \sin 2a + \sin 3a - \dots, \quad \cos a - \cos 2a + \cos 3a - \dots,$$

is to integrate the remainder term by parts. It is worth while to call attention to this detail, because the authors rightly regard the results as important enough, both intrinsically and in relation to the complex logarithm, to be given without proof rather than omitted.

A fundamental problem of the companion volume is foreshadowed in the assertion that it is because no theory of irrational numbers is assumed to be known that the identity of $\exp y$ with e^y has not been proved for irrational values of y. But without a theory of irrational numbers, what is e? With such a theory, whether the definition is Dedekind's or Russell's, there is no difference between e^2 and $e^{\sqrt{2}}$ in difficulty of conception. The distinction in this respect between rational and irrational indices belongs to the time when an irrational number was thought to require a known sequence to establish its existence, and I am sure that when the authors get down to the details of their volume on analysis they will find that in their definition of a^b where a and b are real, the question of rationality is literally irrelevant. This does not imply that the question will not often be relevant in proofs.

The combination of enterprise and experience which we associate with each of the authors individually reaches a remarkable pitch in this book, and it is to be hoped that a collaboration of which the first result is so happy will long be fertile.

E. H. N.

THE LIBRARY.

160 Castle Hill, Reading.

THE GREENSTREET BEQUEST.

A complete list of the books and pamphlets which have come to the Library in consequence of Mr. Greenstreet's death would almost convert a number of the *Gazette* into a catalogue. Since the accession of this material implies that a *Fourth List* will be prepared at the end of this year, duplication on a rather extravagant scale must be avoided. Here follow details only of some of the more interesting books.

M. G. Agnesi	Traités Elémentaires de Calcul - - - - -	1775
	Translated from Italian.	
C. G. Bachet	Problèmes Plaisants et Délectables (4 : A. Labosne). -	1879
W. W. R. Ball	Essay on *Principia* - - - - - - -	1893
J. Booth	New Geometrical Methods ; II - - - -	1877
	Vol. I was already in the Library.	
A. v. Braunmühl	Geschichte der Trigonometrie ; II - - - - -	1903
	Vol. I would be welcome.	
H. Brocard et T. Lemoyne	Courbes Géométriques Remarquables ; I - -	1919
	The plan is that of an Encyclopaedia ; this volume, *Abaque —Courbe auxiliaire*, is the only one that has appeared.	
R. Burrow	A Restitution of the Treatise of Apollonius on Inclinations. Also the Theory of Gunnery -	1779
	Two unrelated essays published together.	
F. Cajori	History of Mathematical Notations ; I - - -	1928
J. E. Campbell	Continuous Groups - - - - - - -	1903
M. Cantor	Geschichte der Mathematik (4 vols.) {3, 2, 2, 1, rep.} -	1922
L. N. M. Carnot	De la Corrélation des Figures - - - - -	1801
	The imprint is *An ix.*	
M. Chasles	Aperçu Historique (3, *i.e.* 1, 1837, rep.) - - -	1889
J. M. Child	The Early Mathematical Manuscripts of Leibniz -	1920
	Translated from Latin, with notes.	
J. Dee	Private Diary, and Catalogue of Library of Manuscripts - - - - - - - - - -	1842
	Edited by J. O. Halliwell.	
A. De Moivre	Miscellanea Analytica - - - - - - -	1730
A. De Morgan	Book of Almanacs (3 : E. J. Worman) - - -	1907
	Budget of Paradoxes (2 : D. E. Smith) (2 vols.) - -	1915
	Essays on Newton - - - - - -	1914
	Edited by P. E. B. Jourdain.	
	Newton : his Friend and his Niece - - -	1885
	Edited by S. E. De Morgan and A. C. Ranyard.	
J. L. E. Dreyer	History of Planetary Systems - - - - -	1906
P. Duhem	Origines de la Statique (2 vols.) - - -	1905, 1906
L. Euler	Methodus inveniendi Lineas Curvas maximi minimive proprietate gaudentes - - - - -	1744
W. Frend	Algebra - - - - - - - - - -	1796
J. Gow	History of Greek Mathematics - - - - -	1884

I. B. HART	Mechanical Investigations of Da Vinci	1925
T. L. HEATH	Apollonius : Conic Sections	1896
S. HORSLEY	Apollonii *Inclinationum Libri Duo*	1770
C. HUYGENS	Celestial Worlds discover'd (2)	1722
	Brevis assertio systematis Saturnii	1660
V. LÉOTAUD	Examen Circuli Quadraturæ quam Gregorius a S. Vincentio exposuit With a reprint of A. de Lionne's *Currilineorum Contemplatio.*	1654
W. LEYBOURN	Compleat Surveyor (2)	1657
	Panarithmologia (15)	1769
G. LORIA	Spezielle Ebene Kurven Translated from Italian into German by F. Schütte.	1902
E. LUCAS	Récréations Mathématiques (4 vols.) (2, 2, 1, 1)	1891-1896
M. MARIE	Histoire des Sciences Mathématiques et Physiques : 10, 11 *Other of the twelve volumes would be acceptable.*	1887
Y. MIKAMI	Development of Mathematics in China and Japan	1912
I. NEWTON	Opuscula (3 vols.)	1744
	Chronology of Ancient Kingdoms Posthumous, prepared by J. Conduitt.	1728
	Opticks (3)	1721
B. NIEWENGLOWSKI	Géométrie Analytique ; I-II *Completing a set.*	1894, 1895
J. OZANAM	Cursus Mathematicus ; III Translated from French into English, by J. T. Desaguliers. *Vols. I, II, IV are wanted.*	1712
T. RUDD	Euclides *Elements of Geometry* "Whereunto is added, The *Mathematicall Preface* of John Dee."	1651
J. SAGERET	Système du Monde des Chaldéens à Newton	1913
G. SALMON	Analytic Geometry of Three Dimensions (5 : R. A. P. Rogers) ; I *Completing a set.*	1912
L. SCHRÖN	Logarithms (5 : De Morgan)	1865
C. A. SCOTT	Modern Analytical Geometry	1894
T. SIMPSON	Essays on several curious and useful subjects in speculative and mix'd Mathematicks	1740
R. SIMSON	Apollonii *Locorum Planorum Libri Duo*	1749
R. F. SLUSIUS	Mesolabum	1668
D. E. SMITH	History of Mathematics ; I *Vol. II is not in the Library.*	1923
	A Source Book in Mathematics	1929
J. SMITH	Quadrature of the Circle Correspondence with A. De Morgan.	1861
W. SNELLIUS	The Books of Apollonius concerning Determinate Section Reprinted by J. Lawson, with another restoration by W. Wales.	1772
J. H. SWALE	Geometrical Amusements	1821

J. TANNERY — Leçons d'Algèbre et d'Analyse (2 vols.) - - - 1906

P. TANNERY — Mémoires Scientifiques ; I, II - - - - 1912
Sciences exactes dans l'antiquité.
Géométrie Grecque ; I - - - - - - - 1887
ALL PUBLISHED.

I. TODHUNTER — Researches in Calculus of Variations - - - 1871

F. VIETA and M. GHETALDUS
The Books of Apollonius concerning Tangencies (2) 1771
Reprinted by J. Lawson, with Fermat's treatise on spherical tangencies.

E. T. WHITTAKER — History of Theories of Æther and Electricity - 1910

J. WILKINS — Mathematical and Philosophical Works - - - 1708

W. H. YOUNG and G. C. YOUNG
Sets of Points - - - - - - - - - 1906

H. G. ZEUTHEN — Histoire des Mathématiques dans l'Antiquité et le Moyen-Age - - - - - - - - - 1902
Translated from German by J. Mascart.

The Athenian Oracle, with Supplement (4 vols.) - - - - 1703-1716

Athenian Sport - - - - - - - - - - - 1707
"Or Two Thousand Paradoxes merrily argued."

The Commentaries of Proclus on Euclid, with a History of the Restoration of Platonic Theology (2 vols.) - - - - - - - - - 1792
By T. Taylor.

Correspondence of Newton and Cotes - - - - - - - - 1850
Edited by J. Edlestone.

The Diarian Miscellany (5 vols.) - - - - - - - - - 1775
C. Hutton's compilation from the *Ladies' Diary*, 1704-1773. Sometimes called *Hutton's Miscellany*, but not to be confused with Hutton's *Miscellanea Mathematica*.

Letters illustrative of the Progress of Science from the reign of Elizabeth to that of Charles II - - - - - - - - - - - 1841
Edited by J. O. Halliwell.

The Schoolmaster (2 vols.) - - - - - - - - - - - 1836
Essays on practical education, by Ascham, De Morgan, Locke, Milton, and others.

In addition to these books, the bequest includes historical and biographical books by F. Arago, P. Bachmann, W. W. R. Ball (2), J. Bertrand (2), A. Binet, M. Bland (2), E. Boutroux, D. Brewster (2), S. Brodetsky, W. W. Bryant, P. Busco, F. Cajori (2), L. Campbell and W. Garnett, A. M. Clerke (2), J. d'Alembert, J. G. Darboux, S. E. De Morgan (2), J. L. E. Dreyer, P. Duhem, R. T. Glazebrook, R. P. Graves, A. Gray, E. S. Haldane, W. Hallock and H. T. Wade, T. L. Heath (4), J. L. Heiberg, J. F. W. Herschel, E. W. Hobson (2), E. Hopkins, W. T. Jeans, B. Jones, D. Larrett, D. E. Lebon (3), G. C. Lewis, L. Liard, O. J. Lodge, G. Loria, A. Macfarlane (2), A. Maire, T. H. Martin, J. T. Merz, G. Milhaud, G. A. Miller, R. E. Moritz, F. Müller (2), J. Munro, M. Napier, J. Narrien, J. P. Nichol, T. N. Orchard, G. Peacock, A. Philip, E. M. Plunket, A. Rebière, W. W. Rupert, P. Sauerbeck, C. F. Smith, D. E. Smith and L. C. Karpinski, D. E. Smith and Y. Mikami, J. W. N. Sullivan, S. P. Thompson (2), I. Todhunter, C. Tweedie, J. Tyndall, C. Warren, W. Whewell (3), W. Whiston, J. Williams, and other books by A. M. Ampère, D. André, W. S. Andrews, E. F. August, C. Babbage (2), F. Baily, A. Bell, J. B. Biot, K. Bopp, J. L. Boucharlat, E. Brassinne, W. E. Byerly, E. Cahen, A. L. Cauchy, M. Chasles, G. Chaucer, J. M. Child, W. K. Clifford, E. Cocker, J. W. Colenso, H. Cox, D. Cresswell, E. Czuber, J. Darby, G. Darley, P. Delens,

E. B. Elliott, L. Euler (2), J. D. Everett, G. Fazzari, J. Ferguson, P. B. Fischer, A. Fisher, E. Fourrey (2), J. H. Grace and A. Young, G. Green, D. F. Gregory, F. Hack, J. Hann, G. W. Hemming, J. S. Jackson, E. de Jonquières, A. E. Kennelly, F. Latoon, J. Lawson, S. Le Clerc, J. Leslie, G. F. A. de l'Hospital, S. Lhuilier, M. Longfield, E. Mach, P. A. MacMahon, J. Napier, F. W. Newman, J. Ozanam, L. Painvin, B. Powell, Ramchundra, J. Richard, H. Schubert, T. Simpson, R. Simson, J. Smith, B. Sporer, C. R. M. Talbot, J. Tannery, T. Tate (2), J. Thomson (2), W. Walton (3), W. Whewell, E. T. Whittaker, E. Wingate, M. Winter, W. S. B. Woolhouse, R. H. Wright.

In the following list of periodicals imperfect runs due to Mr. Greenstreet have already been supplemented in one or two cases. Information concerning some of these journals is to be found in the article written by R. C. Archibald for the 200th number of the *Gazette* (vol. xiv, p. 379).

Bulletin of the New York Mathematical Society. 1-3 - - 1892-1894
The precursor of the *Bulletin of the American Mathematical Society*. See note in *List I*, p. 45.

American Mathematical Monthly. 1→ - - - - - - 1894→
The set is now complete but for vol. 2, parts 6, 11.

Bulletin de Mathématiques Elémentaires. 1-14 - - - - 1896-1909
The journal ceased after the next volume, which the Library would be glad to buy or to exchange for vols. 1-2, of which there are duplicate copies lacking only vol. 2, no. 14.

Bulletin de Mathématiques Spéciales. 1-6 - - - - 1895-1900
A complete set.

Bulletin des Sciences Mathématiques et Astronomiques. 1-11 ; II, 1-17, 25-38, 46 - - - - - - 1870-1893, 1901-1914, 1922
Volumes are wanted to fill the gaps or to extend the run.

L'Education Mathématique. 1-10 - - - - - - - 1899-1908
Lacks vol. 10, no. 2 ; extensions would be welcome.

L'Enseignement Mathématique. 1→ - - - - - - 1899→
For a volume wanted to complete the set the Library is indebted to the Editors.

Gentleman's Diary. 80, 93, 96-97, 99-100 1820, 1833, 1836-1837, 1839-1840

Gentleman's Mathematical Companion. 1-12, 23-28 - 1798-1809, 1820-1825
The first number was entitled *A Companion to the Gentleman's Diary*. The copy of this number is of the first edition, not possessed by the British Museum. On the change of title see Archibald, p. 392.

L'Intermédiaire des Mathématiciens. 1-27 ; II, 1-4 - - - 1894-1925
A complete set. There was a year's gap between the two series and the resuscitated journal has not survived.

Journal de Mathématiques Elémentaires (Vuibert). 1-32 - - 1877-1908
Lacks vol. 13, nos. 2-18 and vol. 17, no. 6.

Journal de Mathématiques Elémentaires (Bourget). 1-4 - - 1877-1880
continued as

Journal de Mathématiques Elémentaires et Spéciales. 5 - - - 1881
divided to form

Journal de Mathématiques Elémentaires
and

Journal de Mathématiques Spéciales
each proceeding

Sér. II ; 6, 2-5. Sér. III ; 1-5. Sér. IV ; 1-5.

Sér. V ; 4 vols., unnumbered and the last uncompleted - - 1882-1900

Ladies' Diary. 90, 92-94, 96-98, 103-108, 114, 132-134, 136-137 1793-1840
With the *Companion* or *Supplement*, 1798, 1806.

Lady's and Gentleman's Diary. 138-168 - - - - - 1841-1871
Numbered serially with the *Ladies' Diary*.

Liverpool Apollonius. 2 - - - - - - - - - - 1824

Mathematical Questions and Solutions from the *Educational Times*. 1-75; II, 1-29 - - - - - - - - - - - - 1864-1916

The two series complete. The first volume, as issued in 1854, consisted of reprints using the original type of the *Educational Times*. The restriction to narrow columns and uncollated material was removed in the second year, when the words "with many additional Solutions not published in the *Educational Times*" appeared on the title-page. In 1886 the first volume was reprinted in the style of the subsequent volumes; this second edition of the volume is in the set, and there is also a copy of the original edition. This biennial is often called the *Educational Times Reprint*.

Mathematical Questions and Solutions in continuation of the Mathematical Columns of the *Educational Times*. 1-4 - - - - 1916-1918

Two volumes are wanted to complete this set.

The Mathematician (Rutherford and Fenwick). 2-3 - - - - 1847-1850

Two of the three volumes as issued in 1856 from the original sheets.

Mathesis. 1-10; II, 1-10; III, 1-10; IV, 1-2 - - - - - 1881-1912

Founded as a sequel to *Nouvelle Correspondance Mathématique*, of which a set was added to the Library early in the year.

Messenger of Mathematics. II, 1-23, 26-44 - - 1872-1894, 1897-1915

This journal comes to an end with vol. 58, and it is much to be hoped that the set can be completed.

Nouvelles Annales de Mathématiques. 1-20; II, 1-20; III, 1-3, 7-19; IV, 1-16 - - - - - - - - - - - - - - 1842-1916

Now wants III, 4-6, and all from the beginning of the fifth series. The third series was stopped at vol. 19 in order to bring the numbering into line with the date.

Palladium. 1-31 - - - - - - - - - - - 1749-1779

A complete set of Heath's *Palladium*. For details of the various titles see Archibald's article.

Revue de Mathématiques Spéciales. 1-12 - - - - - 1891-1914

Sphinx-Œdipe. 1-7 - - - - - - - - - - - - 1906-1912

With a considerable number of separate parts from later volumes.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. The names of those sending the questions will not be published.

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